Multistage stochastic optimization of power dispatch and multiperiod duality of CVaR

Martin Densing [1] János Mayer [2]

[1]Energy Economics Group, Paul Scherrer Institute, Switzerland

[2]Institute for Business Administration, University of Zurich

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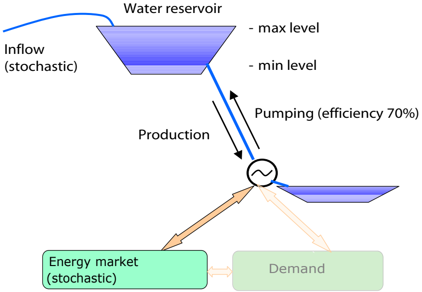
Power plant optimization under risk

Pumped-storage hydropower plant



Optimal control problem: How to dispatch over several months?

Decision criteria: Maximize expected financial profit, under a constraint on financial risk (Mean-risk optimization)

Adaption of decision over time: Multiperiod model (Scenario tree)

Single-period (steady-state) pumped-storage plant model

- No constraint on risk; constraint on water-level in expectation
- $S \in L^1_+(\Omega, \mathcal{F}, \mathbb{P})$ electricity spot price (EUR/MWh), continuous df
- $u^\pm \colon \mathbb{R}_+ \to \mathbb{R}_+$ ctrl-functions, $u^\pm(S)$: turbined/pumped water (MWh)
- $c \in (0,1)$: efficiency of pumping
- Maximal capacity, initial-, minimal-water-level: $u^+_{\max} > l_0 - l_{\min} > 0$

$$\max_{u^\pm} \mathbb{E}\Big[S u^+(S) - \frac{1}{c} S u^-(S) \Big],$$

$$\text{s.t.} \begin{cases} l_0 - \mathbb{E}\big[u^+(S) - u^-(S)\big] \geq l_{\min}, \\ 0 \leq u^\pm(s) \leq u^\pm_{\max} \quad \forall s \in \mathbb{R}_+. \end{cases}$$

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- Optimal solution: $\hat{u}^+(S) = u^+_{\max} 1_{\{S \geq q\}}, \quad \hat{u}^-(S) = u^-_{\max} 1_{\{S \leq cq\}}$

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- Optional solution: see below

- Optimal solution: $\hat{u}^+(S) = u^+_{\max} 1_{\{S \geq q\}}, \quad \hat{u}^-(S) = u^-_{\max} 1_{\{S \leq cq\}}$
- $u^-_{\max} = 0 \Rightarrow$ Optimal objective value $= -(l_0 - l_{\min})\mathrm{CVaR}_{\frac{l_0 - l_{\min}}{u^+_{\max}}}[-S]$
- $c = 1 \Rightarrow$ Newsvendor problem

$\to$ multiperiod extensions possible

Issue: Hourly variations of electricity price

- Generally, more detailed models are only numerically solvable on a scenario tree.
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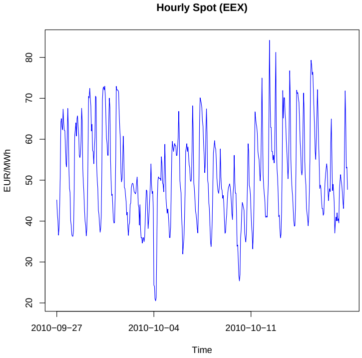


→ Multi-stage stochastic programming problem on hourly time scale
→ Curse of dimensionality: Mid- and long-term models not solvable

Ansatz for multi-stage SP for hydro-power plants

Dynamics of each quantity is modeled on time scale where relative changes are sufficiently large (from the application point of view):

Small time scale: "Hourly"

- Dispatch decisions for production/pumping
- Hourly to weekly variations of spot price
- Hourly water inflow (depends on weather)

Large time scale: "Monthly"

- State of the plant, i.e. relative variations of
  - Cumulative profit-loss
  - Water level of reservoir
- Seasonal variations of spot price
- Seasonal water inflow (winter vs. summer)

Mean-risk optimization model of hydropower plant

- Monthly time steps for the evolution of water-level and profit: $t = 0, 1, \ldots, T$
- Electricity price: $(S_{t+\frac{h}{H}})_{h=1,\ldots,H}$, changes hourly
- $u_t^{\pm}(S_{t+\frac{h}{H}}, \ldots) \in \mathbb{R}_+$: production/pumping control variables

$$\max_{0 \le u^{\pm} \le u^{\pm}_{\max}} \mathbb{E}[X_T],$$

$$\text{s.t.} \begin{cases} R_0^{(X_0,\ldots,X_T)} \ge \rho_{\min}, \\ (L_t, P_t) \in \mathcal{X}_t, \quad t = 0, \ldots, T. \end{cases}$$

- L_t: water level (feasible in every scenario, monthly stochastic inflow)
- P_t: cumulative profit-and-loss
- X_t: production value := P_t + weight-factor × expected usable water
- $R_0^{(X_0,\ldots,X_T)} \in \mathbb{R}$: risk-adjusted value

Hourly time scale replaced by occupation times

- Continuous time. State equations of profit-and-loss and water-level over a month, formally with suitable $f : \mathbb{R} \to \mathbb{R}$:

$$\int_t^{t+1} f(S_{t'})\, dt' = \int_0^{\infty} f(s)\, dF_{t+1}(s), \quad F_{t+1}(s) := \int_t^{t+1} 1_{\{S_t \le s\}} dt',$$

  Stieltjes integral w.r.t. occupation time at level s: $F_{t+1}(s)$.

- Discrete time. discrete price levels $s_0 < s_1 < \cdots < s_N$,

$$\sum_{h=1}^{H} f\left(S_{t+\frac{h}{H}}\right) \approx H \sum_{i=1}^{N} f(\bar{s}_i)\Big(F_{t+1}(s_i) - F_{t+1}(s_{i-1})\Big), \quad \bar{s}_i \in (s_{i-1}, s_i),$$

$$F_{t+1}(s) := \frac{1}{H} \sum_{h=1}^{H} 1_{\{S_{t+\frac{h}{H}} \le s\}} \left(= \begin{array}{l} \text{Fraction of hours where} \\ \text{price} \le s \end{array} \right).$$

Futures contracts: modeled with occupation times

'Phelix' future (EEX): Exchange fixed with floating price during a period.

- Assumptions:
  - period of time: $[t, t+1)$
  - cash settled
  - position amount: p (MW), initially fixed (hedging position)
- Profit-and-loss:

$$P^{\text{fut}} = p \sum_{h=1}^{H} \left(q - S_{t+\frac{h}{H}}\right),$$

q (EUR/MWh): initially contracted future price.

Approximation with occupation times of electricity price:

$$P^{\text{fut}} \approx pH\Big(q - \sum_{i=1}^{N} \bar{s}_i \left(F_{t+1}(s_i) - F_{t+1}(s_{i-1})\right)\Big).$$

Demand: modeled with occupation times

$(D_{t+\frac{h}{H}})_{h=1,2\ldots}$: hourly stochastic demand process; c: retail selling price.
Profit-and-loss over a time period:

$$P_t^{\text{dem}} = \sum_{h=1}^{H} D_{t+\frac{h}{H}} \big(c - S_{t+\frac{h}{H}}\big).$$

Approximation:

$$D_{t+\frac{h}{H}} \approx \sum_{i=1}^{M} \bar{d}_i 1_{\{d_{i-1} < D_{t+\frac{h}{H}} \leq d_i\}}, \quad d_0 < d_1 \cdots < d_M,\ \bar{d}_i \in (d_{i-1}, d_i).$$

$$\rightarrow \quad P_t^{\text{dem}} \approx H \sum_{i,j=1}^{N,M} \bar{d}_j (c - \bar{s}_i) \Big(F_{t+1}(s_i, d_j) - F_{t+1}(s_{i-1}, d_{j-1}) \Big),$$

with joint price-demand occupation time:

$$F_{t+1}(s,d) := \frac{1}{H} \sum_{h=1}^{H} 1_{\{S_{t+\frac{h}{H}} \leq s,\ D_{t+\frac{h}{H}} \leq d\}}.$$

Occupation times of Ornstein–Uhlenbeck (O–U) process

Goal: Dimensional reduction of stochastic vector of occupation times, $(F_t(s_0), \dots, F_t(s_N))$ with N large: Principal Component Analysis (PCA).

Widely adopted (sub-)model of a spot price process $(S_{t'})_{t' \geq 0}$ in continuous time is the O–U process:

$$dS_{t'} = -\mu S_{t'}\, dt' + \sigma dW_{t'}, \qquad t' \in \mathbb{R}_+,$$

or standardized O–U process (after scaling of value and time):

$$dX_t = -X_t\, dt + dW_t, \qquad t \in \mathbb{R}_+.$$

Covariance of occupation time $F(\cdot) := \frac{1}{\tau} \int_0^\tau 1_{\{X_t \leq \cdot\}} dt$ to levels $x_1, x_2 \in \mathbb{R}$:

$$COV\big[F(x_1), F(x_2)\big] := \mathbb{E}\big[F(x_1)F(x_2)\big] - \mathbb{E}\big[F(x_1)\big]\mathbb{E}\big[F(x_2)\big].$$

Covariances of occupation times of O–U process

Ansatz: Moment-generating function in $s_1, s_2 \in \mathbb{R}_+$ [Feynman–Kac, 1949],

$$\mathbb{E}[e^{-s_1\tau F(x_1)-s_2\tau F(x_2)}] = e^{-\tau E(s_1,s_2)},$$

$E(s_1, s_2)$ is lowest eigenvalue of perturbed quantum harmonic oscillator:

$$\mathcal{H} = 1/4\big(\partial^2/\partial x^2 + x^2\big) + s_1 1_{\{x\le x_1\}} + s_2 1_{\{x\le x_2\}}.$$

Perturbation analysis yields, $\tau \to \infty$:

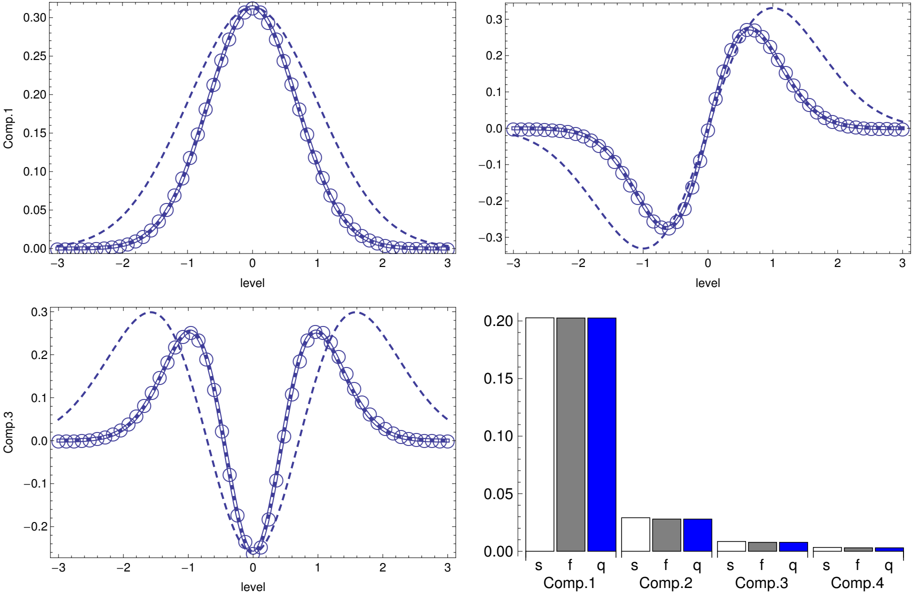
$$COV\big[F(x_1), F(x_2)\big] = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}(n+1)(n+1)!}H_n(x_1)e^{-x_1^2}H_n(x_2)e^{-x_2^2},$$

$H_n(\cdot)$ nth Hermite polynomial.
A functional principal component f with variance λ given by eigenvalue equation:

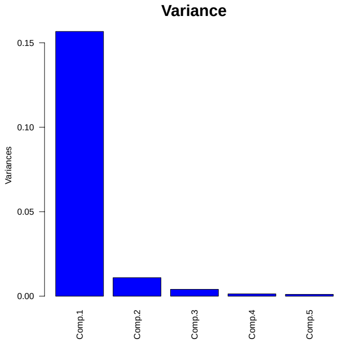
$$\int_{-\infty}^{\infty} COV\big[F(x_1), F(x_2)\big] f(x_2)\, dx_2 = \lambda f(x_1).$$

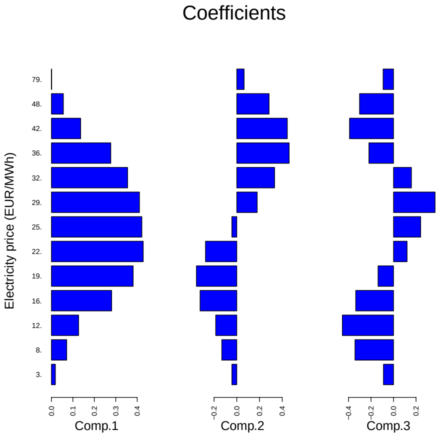
Fredholm equation of second kind $\to$ numerical solutions

Functional PCA of occupation times of O–U process



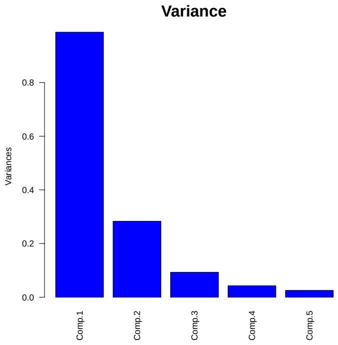
Monthly occupation times from hourly electricity prices (EEX market, 2003-2005). Principal component analysis:

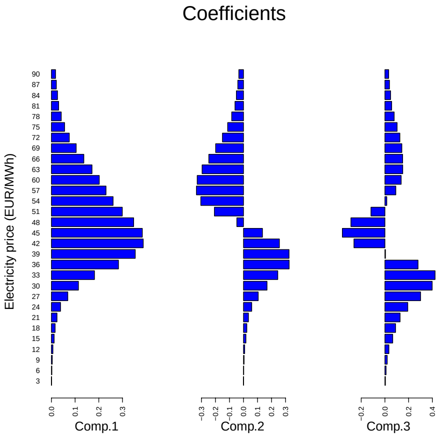




PCA of occupation times of electricity price (cont.)

NordPool market, 2008-2010, weekly time steps, constant price steps:





→ Principal-component factor model

Mean-risk optimization with risk-adjusted values

Risk constraint for the financial value process $(X_0, \ldots, X_T)$ by a lower bound on a coherent risk-adjusted value $R_0^{(X_0,\ldots,X_T)}$:

$$\max \mathbb{E}[X_T], \quad \text{s.t.} \begin{cases} R_0^{(X_0,\ldots,X_T)} \geq \rho_{\min}, \\ \ldots . \end{cases} \quad \overset{\text{equiv.}}{\iff} \quad \max \mathbb{E}[X_T], \quad \text{s.t.} \begin{cases} r_0 \geq \rho_{\min}, \\ \mathbf{A}(X_0, \ldots, X_T, r_0, \ldots)^\top \leq \mathbf{0}, \\ r_0 \in \mathbb{R}, \qquad \ldots . \end{cases}$$

1. Example:

Conditional Value-at-Risk [Acerbi and Tasche, 2001; Pflug, 2007]

$(\Omega, \mathbb{P}, \mathcal{F})$; $X_T \in L^\infty$; $\alpha \in (0,1)$:

$$\text{CVaR}^\alpha[X_T] := \min_{\left\{\mathbb{Q}:\, \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\alpha}\right\}} \mathbb{E}_{\mathbb{Q}}[X_T] = \max_{q \in \mathbb{R}} \left(q - \frac{1}{\alpha} \mathbb{E}[(q - X_T)^+] \right).$$

2. Example: Time consistent risk-adjusted values

Finite setting: Scenario tree $\leftrightarrow$ Filtration $(\mathcal{F}_t)_{t=0,\dots,T}$

Recursively defined risk-adjusted value process [Artzner et al., 2007]:

$$R_t^{(X_0,\dots,X_T)} := \begin{cases} X_T, & \text{if } t = T, \\ \min\Big(X_t, \min\limits_{\mathbb{Q}\in\mathcal{P}^\alpha} \mathbb{E}_{\mathbb{Q}}[R_{t+1}^{(X_0,\dots,X_T)} | \mathcal{F}_t]\Big), & \text{if } t = 0,\dots,T-1. \end{cases}$$

Proposition (Densing and Mayer, 2012)

If $\mathcal{P}^\alpha$ is suitably chosen, then

$$R_0^{(X_0,\dots,X_T)} = \max\ R_0,$$

$$s.t. \begin{cases} R_t \leq X_t, & t = 0,\dots,T, \\ R_t \leq Q_t - \frac{1}{\alpha}\mathbb{E}[Z_{t+1}|\mathcal{F}_t], & t = 0,\dots,T-1, \\ Z_t \geq Q_{t-1} - R_t, \quad Z_t \geq 0, & t = 1,\dots,T, \\ R_t, Q_t, Z_t \quad \mathcal{F}_t\text{-measurable } \forall t. \end{cases}$$

3. Example: Stopped CVaR

- *Stopping time* $\tau : \Omega \to \{0, 1, \ldots, T\}$, with $\{\tau = t\} \in \mathcal{F}_t \; \forall t$
- $X_\tau(\omega) := X_{\tau(\omega)}(\omega)$
- $\mathcal{T}$: set of stopping times

Proposition (Densing, 2012, forthcoming)

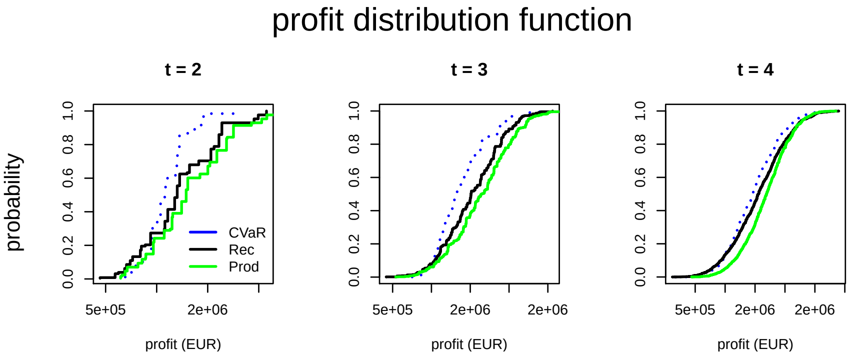
$$\min_{\tau \in \mathcal{T}} CVaR^\alpha[X_\tau] = \max \; R_0,$$

$$s.t. \begin{cases} R_t \leq \mathbb{E}[R_{t+1} | \mathcal{F}_t], & t = 0, \ldots, T-1, \\ R_t \leq q - \frac{1}{\alpha} Z_t, & t = 0, \ldots, T, \\ Z_t \geq q - X_t, & t = 1, \ldots, T, \\ Z_t \geq 0, & t = 1, \ldots, T, \\ q \in \mathbb{R}, \qquad R_t, Z_t & \mathcal{F}_t\text{-measurable } \forall t. \end{cases}$$

Profit distribution over time (5-stage tree)

Model-runs with different risk measurement:

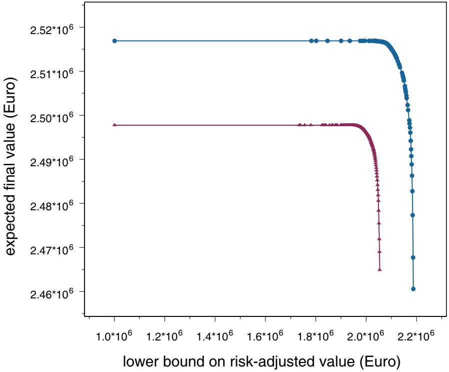
- CVaR (on final values)
- Extension of CVaR recursively
- Extension of CVaR with stopping times



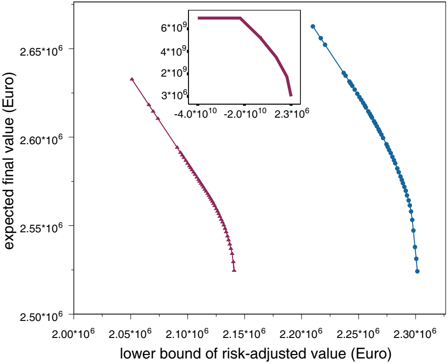
→ differences in intermediate times for cumulated profit-and-loss

Mean-risk frontier: effect of futures, of stochastic inflow

Investment in futures prohibited:



Futures are allowed:



—: Deterministic water inflow into reservoir

—: Stochastic water inflow ($\rightarrow$ reduction of the expected final value)

Selected references

Energy optimization with SLP (abbreviated titles):

- **Eichhorn, Heitsch, Römisch:** SP of elect. portfolios (2010)
- **Güssow, Frauendorfer:** MSP for power systems (2002)
- **Hochreiter, Wozabal:** MSP for risk-optimal elect. portfolios (2010)
- **Kovacevic, Wozabal:** Semiparam. model for elect. prices (2012)
- **Vespucci et al.:** SP for the daily coordination of pumped storage and wind (2012)
- **Densing:** Occupation Times of O–U process (Physica A, 2012)
- **Densing:** SP for Hydro-Power Dispatch Planning: Exact Solutions and Occupation Times (submitted)

Coherent Risk Measurement:

- **Pflug, Pichler:** Decomposition of Risk Measures (2011)
- **Densing, Mayer:** MSP with Time-Consistent Risk Constraints (OR 2011 Proc., 2012)

Conclusion

- Simple models of pumped-storage plants are exactly solvable (bang-bang solutions)
- Hourly trading in mid- and long-term models possible ($\leftarrow$ principal component analysis of occupation times of Ornstein–Uhlenbeck process)
- Multiperiod constraints on risk can be incorporated