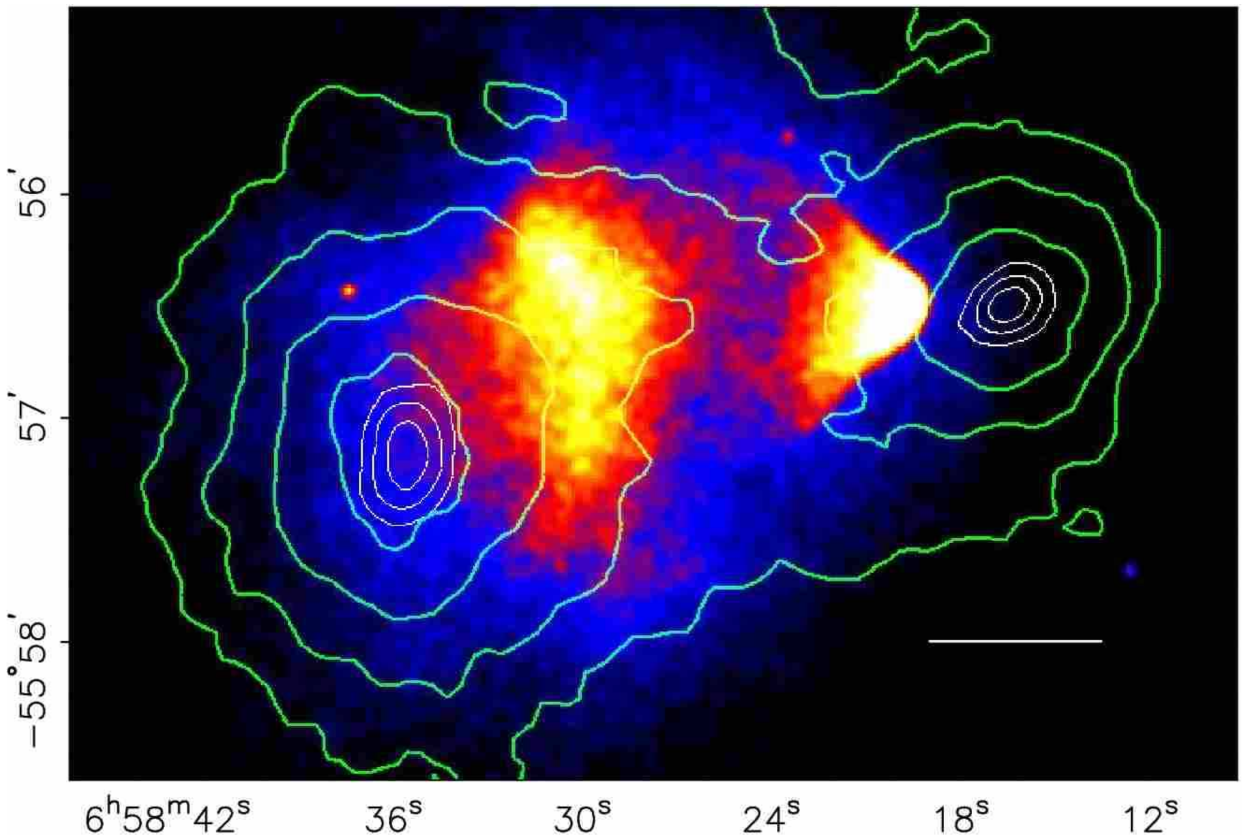
On the quantum statistical physics of dark matter freeze-out[1,2]

Mikko Laine

University of Bern, Switzerland

[1] Based partly on collaborations with Simone Biondini and Seyong Kim.
[2] Supported by the SNF under grant 200020-168988.

There is “dark matter”



[3] D. Clowe *et al*, *A direct empirical proof of the existence of DM,* astro-ph/0608407.

What is dark matter?

Yet to be discovered particles? Basic requirements:

- not visible $\Rightarrow$ electrically neutral
- around long ago & still today $\Rightarrow$ stable or very long-lived
- correct structure formation long ago $\Rightarrow$ rather heavy

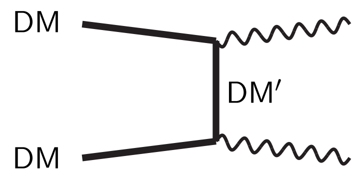
Known particles (well described by the Glashow-Weinberg-Salam "Standard Model") fail to satisfy these requirements.

(An extension by keV sterile neutrinos might help. An extension by μeV axions is quite popular. Primordial black holes may help, but they cannot be generated within the Standard Model.)

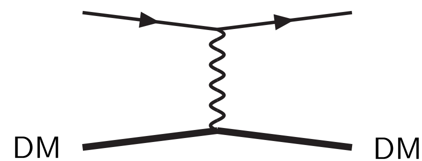
In this talk: "WIMP paradigm"

Postulate the existence of Weakly Interacting Massive Particles ("heavy neutrinos") which cannot decay and are thus stable.

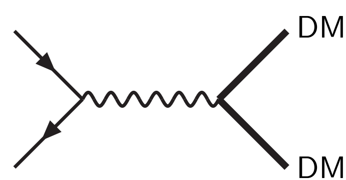
"Indirect detection" from galactic center:



"Direct detection" by nuclear recoil:



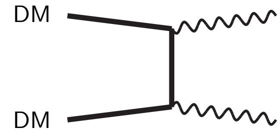
"Collider search" through missing energy:



Text-book WIMP is in trouble

Lee-Weinberg equation[4] ($n =$ number density, $H =$ Hubble rate)

$$(\partial_t + 3H)n = -\langle \sigma v_{\text{rel}} \rangle \left(n^2 - n_{\text{eq}}^2\right) .$$



Entropy conservation (part of Einstein equations)

$$(\partial_t + 3H)s = 0 .$$

Combining the two we get

$$\partial_t \left(\frac{n}{s}\right) = -\langle \sigma v_{\text{rel}} \rangle \frac{\left(n^2 - n_{\text{eq}}^2\right)}{s} .$$

[4] B.W. Lee and S. Weinberg, *Cosmological Lower Bound on Heavy Neutrino Masses*, Phys. Rev. Lett. 39 (1977) 165.

Insert relation of time and temperature ($c =$ heat capacity):

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{3Hs}{c} \,, \quad \partial_t = \frac{\mathrm{d}T}{\mathrm{d}t}\,\partial_T \;.$$

Defining a "yield parameter" through $Y \equiv n/s$ we get

$$\partial_T Y = \frac{c\langle \sigma v_{\text{rel}} \rangle}{3H}\,(Y^2 - Y_{\text{eq}}^2) \;.$$

Inserting $c \sim T^2$, $\langle \sigma v_{\text{rel}} \rangle \sim \alpha^2/M^2$, $H \sim T^2/m_{\text{Pl}}$, where M and α denote the WIMP mass and coupling, we get

$$\partial_T Y \sim \frac{m_{\text{Pl}}\alpha^2}{M^2}(Y^2 - Y_{\text{eq}}^2) \;.$$

The equilibrium number density is a known function of T, M:

$$n_{\text{eq}} \propto \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{e^{\sqrt{p^2+M^2}/T} \pm 1} \approx \left(\frac{MT}{2\pi}\right)^{3/2} e^{-M/T} \,.$$

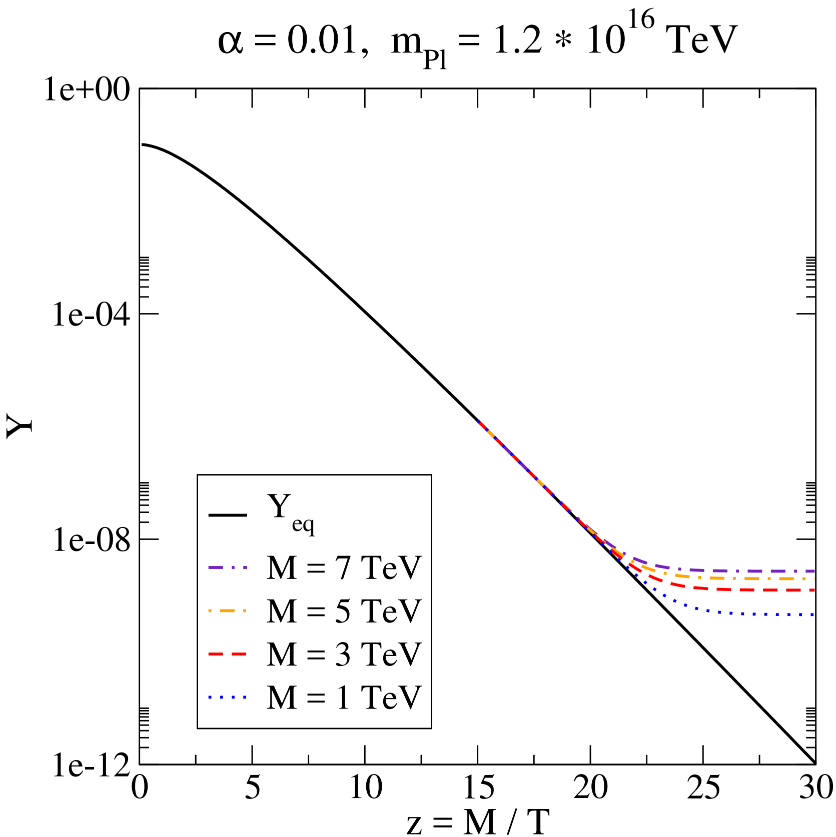
The entropy density is dominated by massless modes: $s \propto T^3$.

Linearize around equilibrium ($Y = Y_{\text{eq}} + \delta Y$, $\delta Y = Y - Y_{\text{eq}}$):

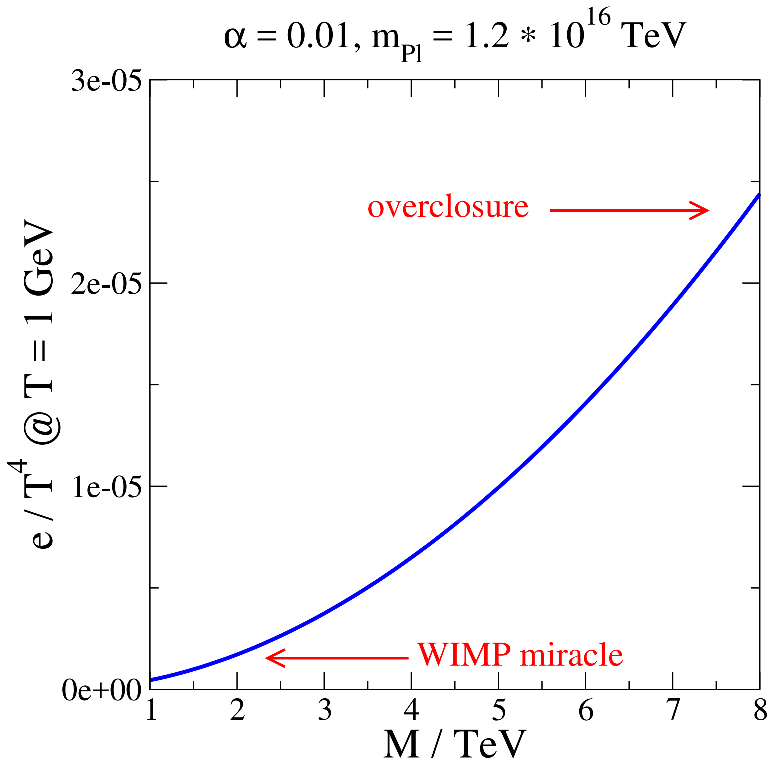
$$z \equiv \frac{M}{T} \;\Rightarrow\; \boxed{Y'(z) \sim -\frac{m_{\text{Pl}}\alpha^2}{M} \frac{Y_{\text{eq}}(z)[Y(z) - Y_{\text{eq}}(z)]}{z^2}} \,.$$

The differential equation has a "thermal fixed point" at $Y(z) = Y_{\text{eq}}(z)$ but cannot keep close to it for $Y_{\text{eq}}(z) \ll 1$.

Numerical solution shows a “freeze-out”:



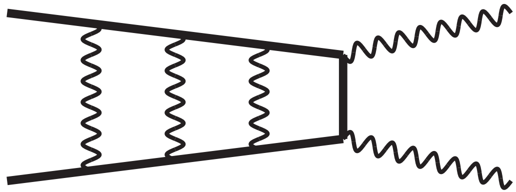
Final energy density ($e \equiv Mn$) compared to radiation $\sim T^4$:



LHC pushes up M, so there is a danger "overclosure"!

Could increased $\langle \sigma v_{\text{rel}} \rangle$ help?

Much discussed:[5] "Sommerfeld effect"[6]:



$$\langle \sigma v_{\text{rel}} \rangle \longrightarrow \langle \sigma_{\text{tree}}\, v_{\text{rel}}\, S(v_{\text{rel}}) \rangle \ .$$

For attractive Coulomb-like interaction,

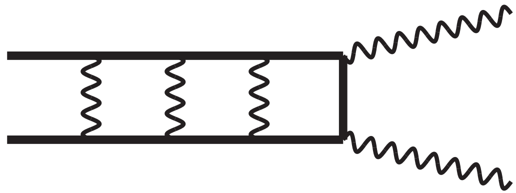
$$S(v_{\text{rel}}) \sim \frac{\alpha}{v_{\text{rel}}} \quad \text{for} \quad v_{\text{rel}} \lesssim \alpha \ .$$

---

[5] e.g. J. Hisano, S. Matsumoto, M. Nagai, O. Saito and M. Senami, *Non-perturbative effect on thermal relic abundance of dark matter*, hep-ph/0610249; J.L. Feng, M. Kaplinghat and H.-B. Yu, *Sommerfeld Enhancements for Thermal Relic Dark Matter*, 1005.4678.

[6] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics, Non-Relativistic Theory*, Third Edition, §136; V. Fadin, V. Khoze and T. Sjöstrand, *On the threshold behavior of heavy top production*, Z. Phys. C 48 (1990) 613.

More recent:[7] bound states ("WIMPonium"):



$$M_{\text{bound}} = 2M - \Delta E \;\Rightarrow\; e^{-M_{\text{bound}}/T} > e^{-2M/T} \;.$$

This is quantum mechanics in a statistical background.

7 e.g. B. von Harling and K. Petraki, *Bound-state formation for thermal relic dark matter and unitarity*, 1407.7874.

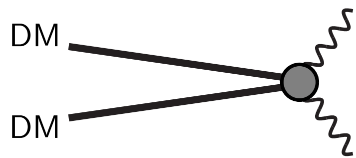
Some quantum statistical physics

Particles in the initial state: most energy is carried by mass.

$$E_{\text{rest}} \sim 2M \; , \quad E_{\text{kin}} \sim \frac{k^2}{2M} \sim T \; .$$

Particles in the final state: all energy is carried by momentum.

$$E_{\text{kin}} \sim 2k \sim 2M \quad \Rightarrow \quad \Delta x \sim \frac{1}{k} \sim \frac{1}{M} \ll \frac{1}{T} \; .$$

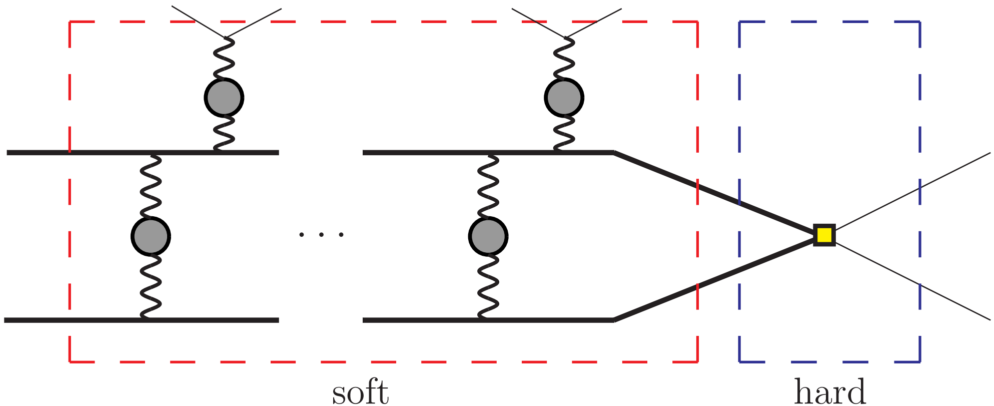


Therefore the "hard" annihilation process is *local*.[8]

---

[8] e.g. G.T. Bodwin, E. Braaten and G.P. Lepage, *Rigorous QCD analysis of inclusive annihilation and production of heavy quarkonium*, hep-ph/9407339; L.S. Brown and R.F. Sawyer, *Nuclear reaction rates in a plasma*, astro-ph/9610256.

But before the annihilation there are “soft” initial-state effects:



A "linear response" analysis shows that

$$\langle \sigma v_{\text{rel}} \rangle \sim \frac{\alpha^2}{M^2 n_{\text{eq}}^2} \underbrace{\frac{1}{Z} \sum_{m,n} e^{-E_m/T} \langle m | \phi^\dagger \phi^\dagger \overbrace{|n\rangle\langle n|}^{\Rightarrow \mathbb{1}} \phi\phi | m \rangle}_{=:\langle \mathcal{O}^\dagger(0)\mathcal{O}(0)\rangle_T} \; .$$

Here $|m\rangle$ are eigenstates containing a DM-DM pair, and $\phi\phi$ annihilates the DM-DM pair.

The DM-DM problem can be reduced to a 1-body problem:

$$E_m \quad =: \quad E' + \underbrace{\left[2M + \frac{k^2}{4M}\right]}_{\text{center-of-mass energy}} \quad .$$

Carrying out the integral over k we are left with

$$\langle \mathcal{O}^\dagger(0)\mathcal{O}(0)\rangle_T = e^{-2M/T}\left(\frac{MT}{\pi}\right)^{3/2}\int_{-\Lambda}^{\infty}\frac{\mathrm{d}E'}{\pi}e^{-E'/T}\tilde{\rho}(E') \; .$$

The “spectral function” $\tilde{\rho}(E')$ represents the solution of a Schrödinger equation for a Green's function.

$$\begin{aligned}\left[H_T - i\,\Gamma_T(r) - E'\right] G(E';\mathbf{r},\mathbf{r}') &= \delta^{(3)}(\mathbf{r}-\mathbf{r}')\ , \\ \lim_{\mathbf{r},\mathbf{r}'\to\mathbf{0}} \mathrm{Im}\, G(E';\mathbf{r},\mathbf{r}') &= \tilde{\rho}(E')\ .\end{aligned}$$

Here the Hamiltonian has a standard from

$$H_T = -\frac{\nabla_{\mathbf{r}}^2}{M} + V_T(r)\ , \quad r = |\mathbf{r}|\ ,$$

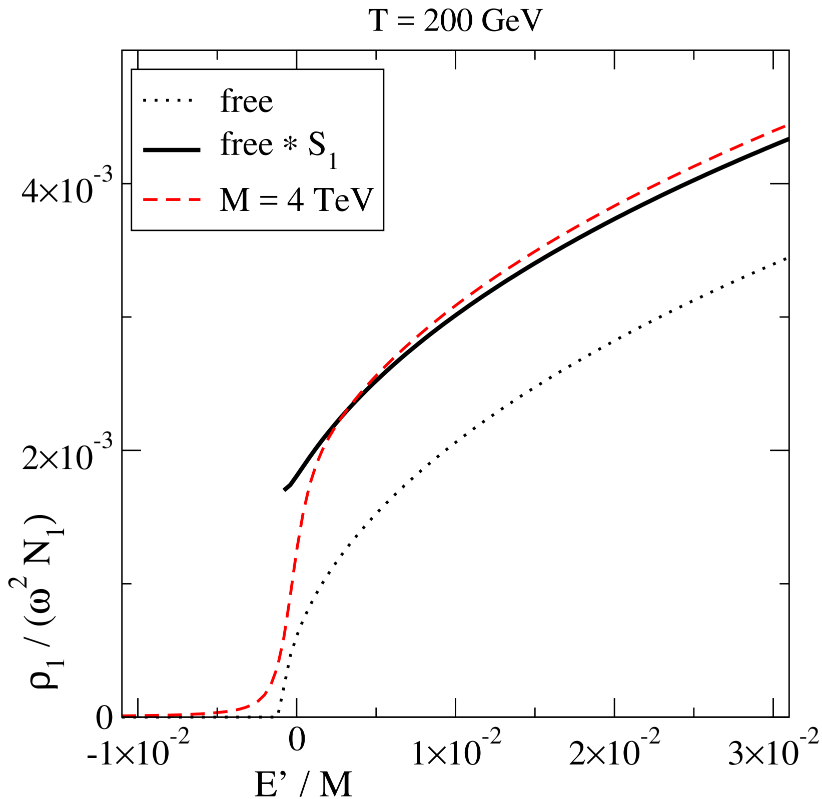
whereas $-i\Gamma_T(r)$ accounts for real scatterings with the plasma.

Procedure in a nutshell

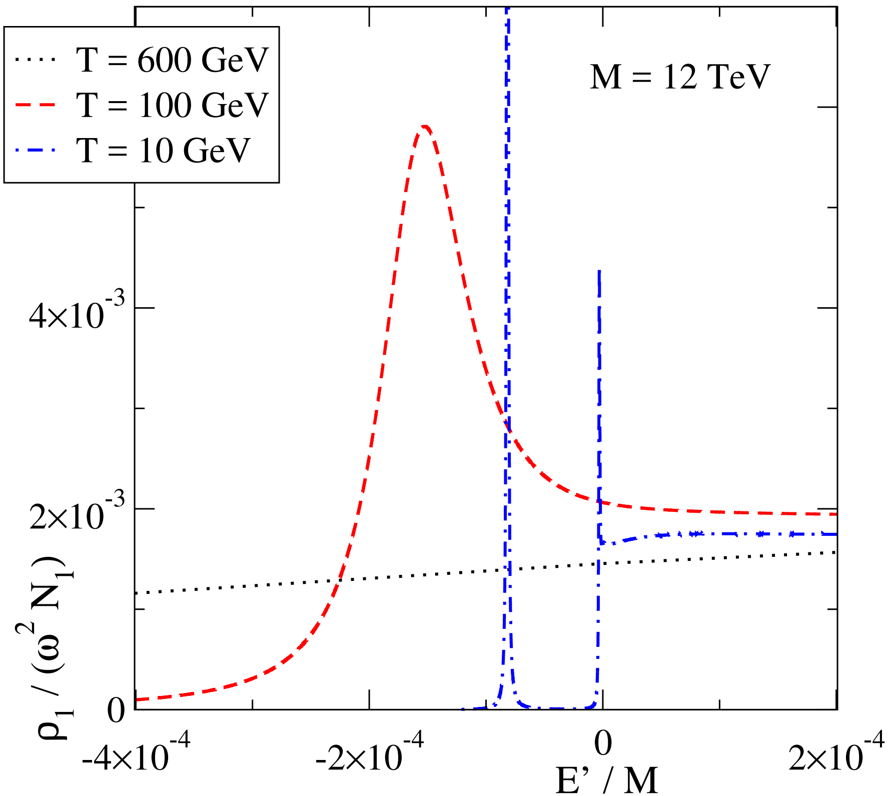
- Compute thermal self-energy of the exchanged particle
- Determine the corresponding time-ordered propagator
- Fourier-transform for potential $V_T(r)$ and width $\Gamma_T(r)$
- Solve for $\tilde{\rho}(E') = \operatorname{Im} G(E'; \mathbf{0}, \mathbf{0})$
- Laplace-transform with weight $e^{-E'/T}$ for $\langle \mathcal{O}^\dagger(0)\mathcal{O}(0)\rangle_T$
- Integrate Lee-Weinberg equation numerically

Results for simple cases

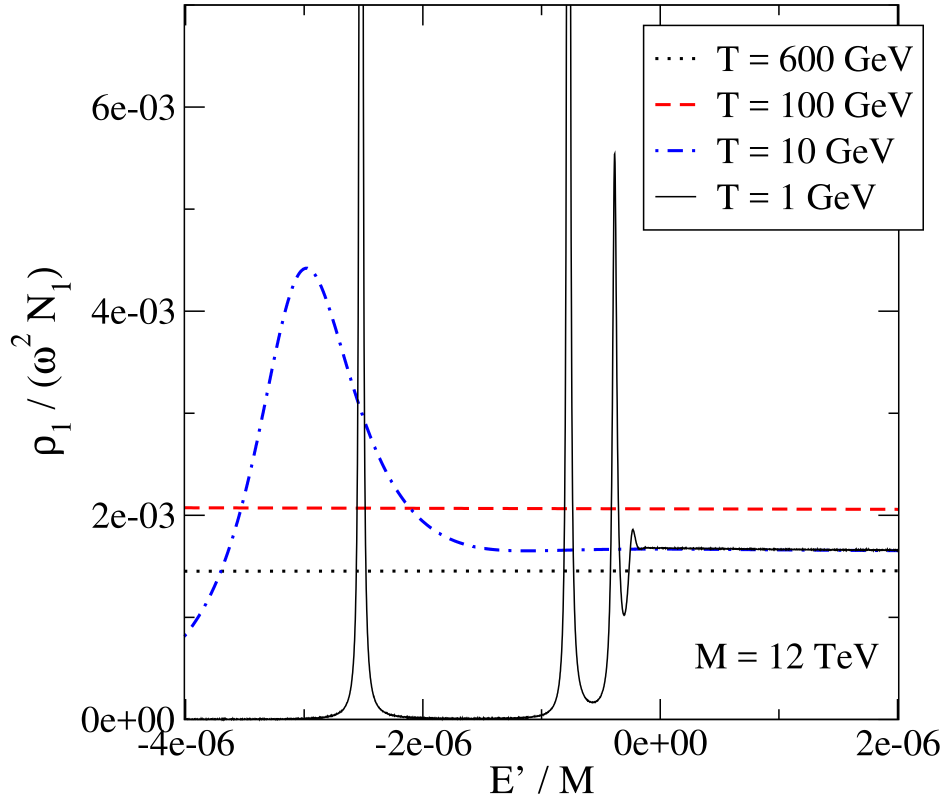
Spectral function for typical WIMPs



It becomes interesting if M is large and T low

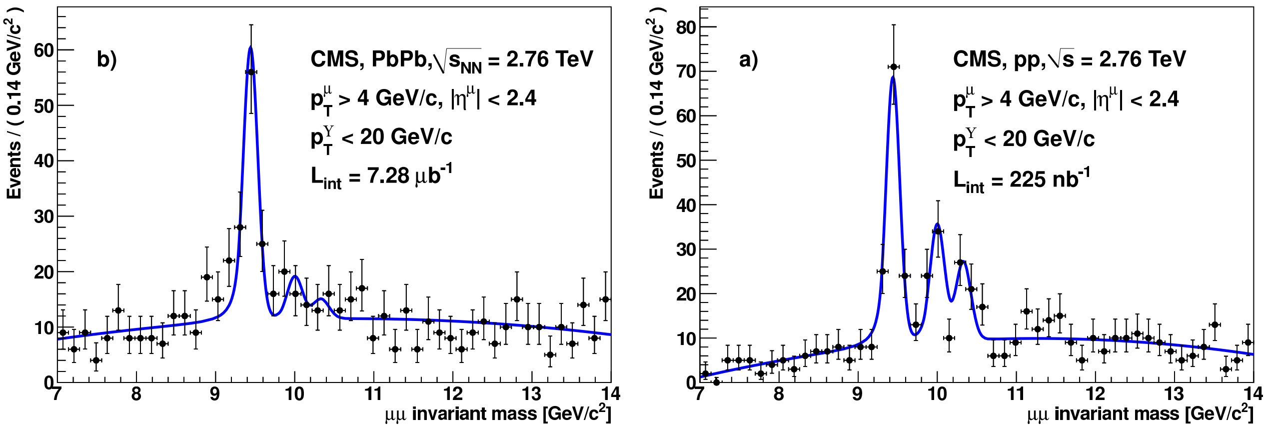


Even more interesting if we decrease T further



A nice relation to heavy ion collision experiments

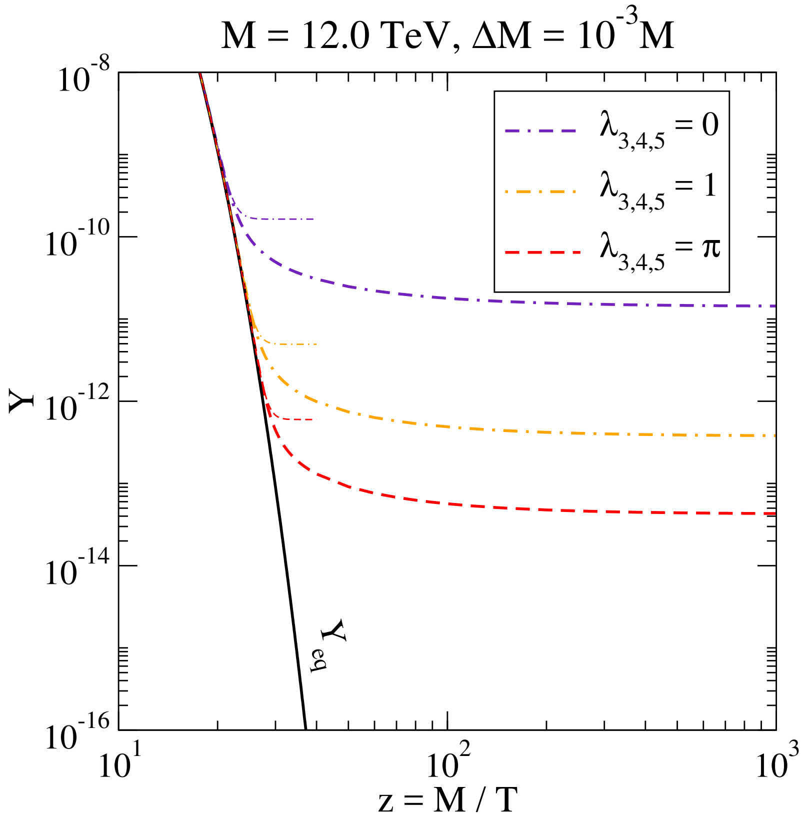
S. Chatrchyan et al. [CMS Collaboration], *Suppression of excited Υ states in PbPb collisions at $\sqrt{s_{NN}} = 2.76$ TeV*, Phys. Rev. Lett. 107 (2011) 052302 [1105.4894].



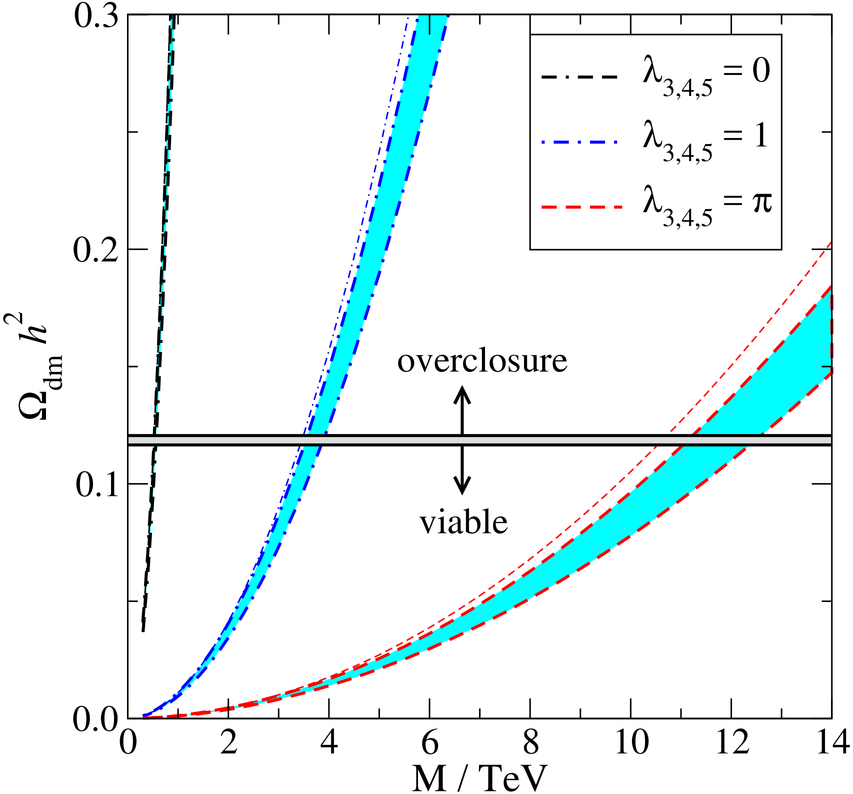
This follows a general pattern previously predicted theoretically.[9]

[9] e.g. F. Karsch, D. Kharzeev and H. Satz, *Sequential charmonium dissociation*, Phys. Lett. B 637 (2006) 75 [hep-ph/0512239].

Dark matter density evolution in the Early Universe



Examples of overclosure bounds

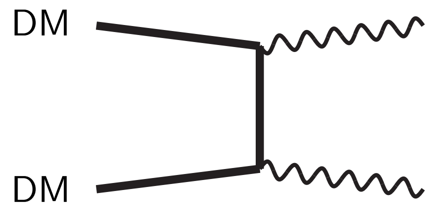


Relation to indirect non-detection

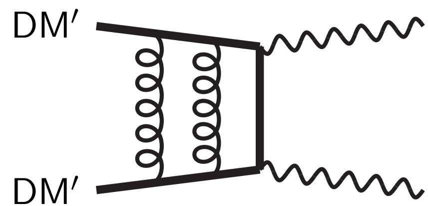
Why is Early Universe not much constrained by present day?

Long ago: $t \sim 10^{-12}$ s, $T \sim 100$ GeV.

DM annihilation:



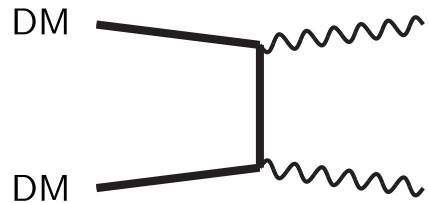
DM' annihilation:



DM $\leftrightarrow$ DM' is in thermal equilibrium $\Rightarrow$ annihilation can proceed through the heavier DM' channel if this is more efficient.

Today: $t \sim 10^{17}$ s, $T \ll$ eV.

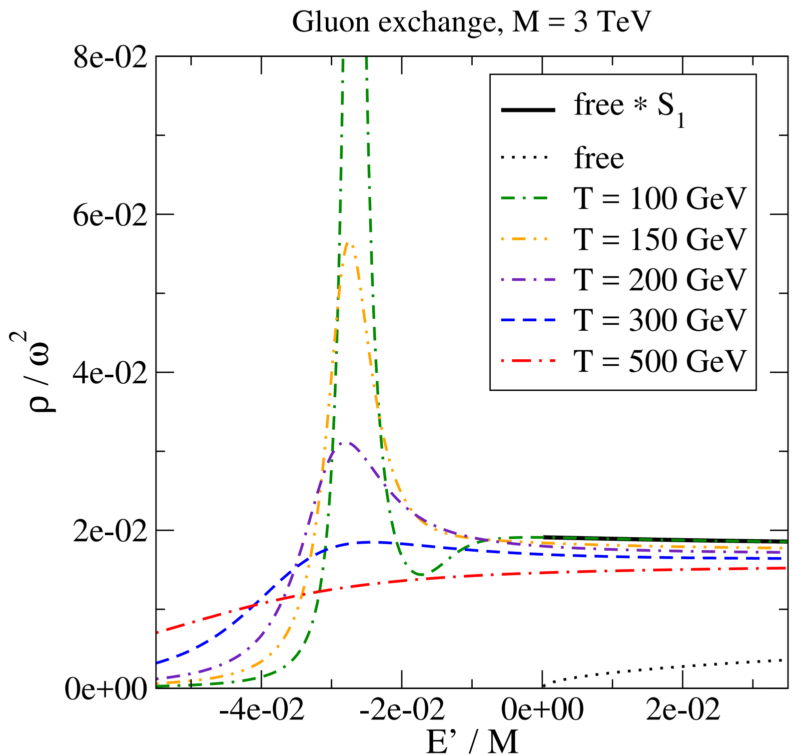
DM annihilation is active:



DM' decayed long ago.

Therefore present day indirect detection constraints cannot directly fix $\langle \sigma v_{\rm rel} \rangle$ relevant for Early Universe cosmology.

If DM’ interacts strongly, there are large effects.[10]



[10] e.g. J. Ellis, F. Luo and K.A. Olive, *Gluino Coannihilation Revisited*, 1503.07142.

Summary

- Apart from model uncertainties, generic dark matter studies also contain theoretical uncertainties.

- In general, both quantum-mechanical effects (bound states, multiple interactions) and statistical physics phenomena (Debye screening, frequent scatterings on plasma particles) play a role.

- Strong interactions may enhance the annihilation rate because of bound states, and this may help to avoid overclosure.

- Model-specific studies are needed for definite conclusions.