3. Muonium and muonium spectroscopy

Muonium, Mu $\equiv \mu^+ e^-$, is a true Hydrogen isotope.

Properties:

Mass: $m_{Mu} = 0.1131\, m_H = 207.77\, m_e$

Reduced Mass: $\bar{m}_{Mu} = 0.9956\, \bar{m}_H = \left(\frac{m_\mu m_e}{m_\mu + m_e}\right) \cong m_e$

Bohr Radius (n=1): $a_{Mu} = 1.0044\, a_0$ $\qquad a_{Mu} = \frac{4\pi\varepsilon_0 \hbar^2}{\bar{m}_{Mu} e^2} = 0.05315$ nm

$(a_0 = 4\pi\varepsilon_0 \frac{\hbar^2}{m_e e^2})$

generally (n-th level): $r_n = n^2 a_{Mu}$

Ionisation energy: $R_{Mu} = 0.9956\, R_y$ $\qquad R_{Mu} = \frac{\bar{m}_{Mu} e^4}{(4\pi\varepsilon_0)^2 2\hbar^2} = 13.54$ eV

$R_y = \frac{m_e e^4}{(4\pi\varepsilon_0)^2 2\hbar^2} = \frac{\alpha^2 m_e c^2}{2}$

for the n-th level: $\frac{R_{Mu}}{n^2}$

Hyperfine coupling: $A_{Mu} = 3.1423\, A_H$ $\qquad A_{Mu} = \frac{2}{3}\mu_0 g_e |\mu_B| g_\mu \mu_B^\mu \frac{1}{\pi\, a_{Mu}{}^3}$

$= h \cdot 4463.3$ MHz

"Nuclear "- gyromagnetic

factor: $\gamma_\mu = 3.18335\gamma_p$ $\qquad \frac{\gamma_\mu}{2\pi} = 13.5534$ kHz / G $= 135.534$ MHz / T

Muonium gyromagnetic factor

in triplett state $(F = 1, M = \pm 1)$: $\gamma_{Mu}^T = 1.0033\, \gamma_H^T$ $\qquad \gamma_{Mu}^T = \frac{1}{2}(\gamma_e - \gamma_\mu) =$

(in weak fields) $\qquad = 2\pi \cdot 1.3944$ MHz/G

$= 102.88\, \gamma_\mu$

A_{Mu} in cgs units: multiply with $\frac{4\pi}{\mu_0} \rightarrow \frac{8\pi}{3}$ instead of 2/3

R_y: Rydberg energy. Ionization energy of a H-Atom with infinitely heavy nucleus.
R_y [eV] = hc R_∞ [m^{-1}]

Muonium is particularly interesting for spectroscopic investigations because:

- Simple, pure leptonic system.
- Only sensitive to weak, electromagnetic interaction, and gravitation.
- μ^+ : point like particle (from scattering experiments → dimension < 10^{-18} m=10^{-3} fm ~ 1/1000 proton radius).

Muonium can be used to test fundamental laws and symmetries and for precision measurements of fundamental parameters.

Examples are measurements of:

- Hyperfine structure → α, $\frac{\mu_\mu}{\mu_p}$, or $\frac{m_\mu}{m_e}$
- Muonium 1s-2s measurements → new determination of fine structure constant α
- Lamb shift ($2S_{1/2}$-$2P_{1/2}$) in Mu not yet precise enough for comparison with theory

Hydrogen

Electron g-2 factor or (g-2)/2=a_e, and $\Delta\nu^H_{hfs}$ are among the best known quantities in physics.

E.g. a_e known to 0.23 ppb → α with 0.32 ppb error.

$\Delta\nu^H_{hfs}$ even known to 0.6 ppt (10^{-12}), but theoretical description is only possible at ppm level because of internal structure of the proton (radius, polarisibility). Similarly for $2S_{1/2}$-$2P_{1/2}$ Lamb shift.

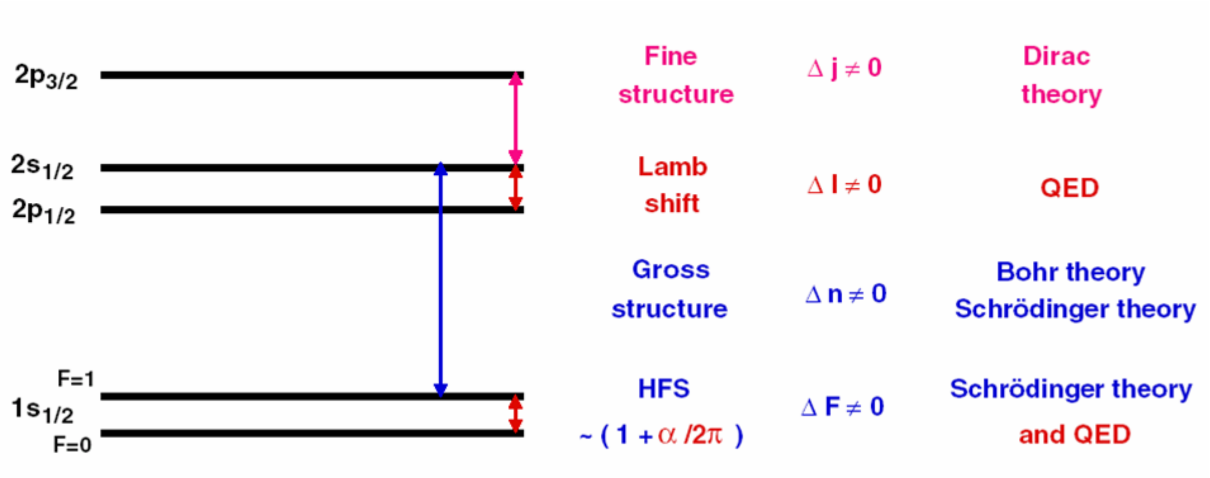
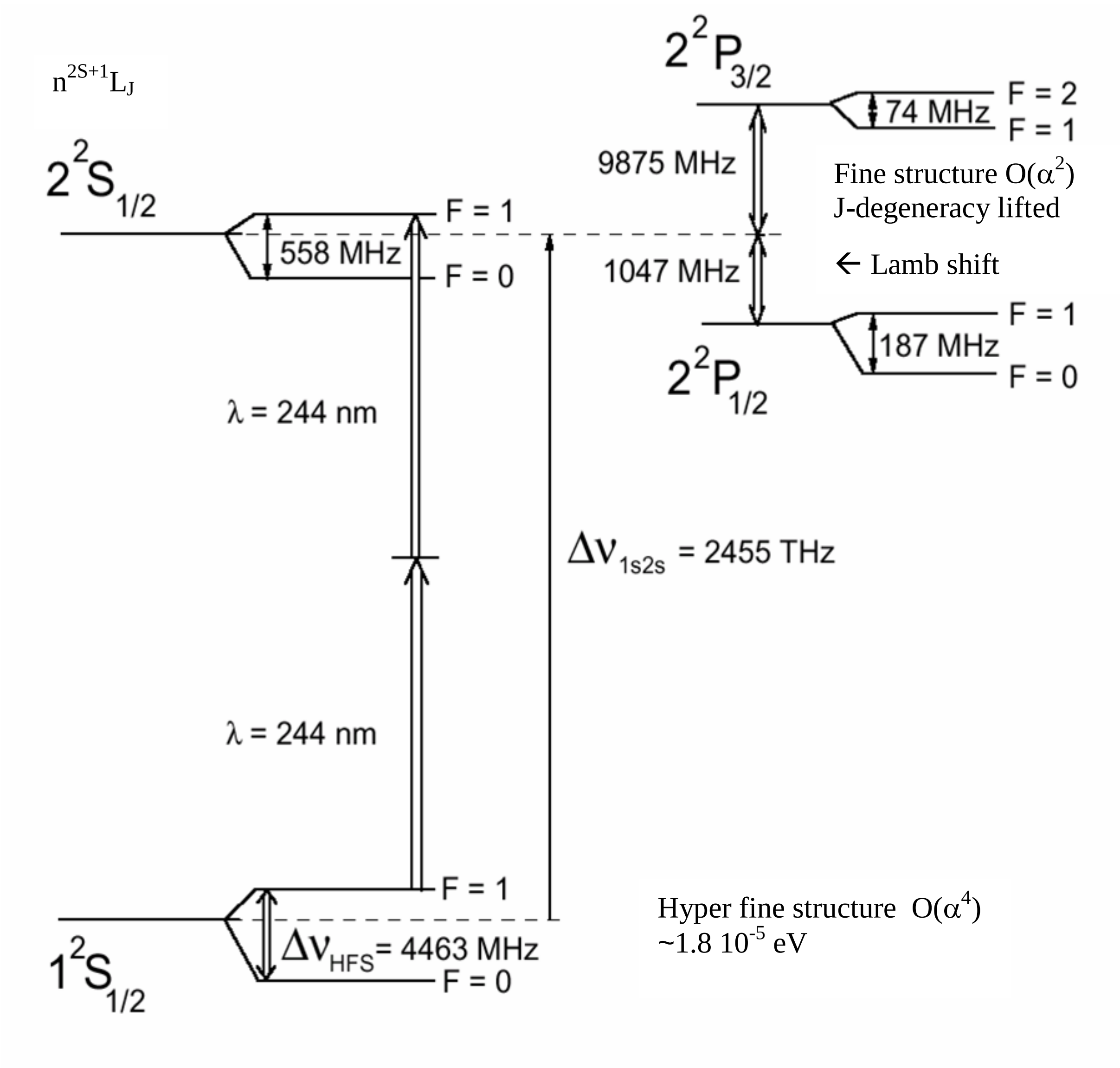


Fig. 3-1: Some low lying energy levels of hydrogen atom (or muonium), not to scale.

$$\vec{J} = \vec{L} + \vec{S}$$
$$\vec{F} = \vec{J} + \vec{I}$$

Energy level of muonium, n=1 und n=2



Life time $\tau_\mu = 2.2\,\mu s$, both ground state and excited state decay with this time constant → from uncertainty relation:

$$\Delta E\tau = 2\hbar, \quad \Delta E = h\Delta\nu_{nat} \rightarrow \Delta\nu_{nat} = \frac{2}{2\pi\tau} \cong 145 \text{ kHz} \quad \text{(natural limit of precision)}$$

3.1 Theory of the energy levels of a muonium atom

The total energy of an electron in a one-electron atom can be expressed in the following way:

$$E_{tot}(n; j; l; F) = E_D(n; j) + E_{RM}(n; j; l) + E_{QED}(n; j; l) + E_{HFS}(n; j; l; F; I) + E_{strong} + E_{weak} + E_{exotic} \quad [3\text{-}1]$$

E_D: Dirac energy for an electron in a point like infinite heavy nucleus with charge Z , which creates a potential V = – Zq / r . The Dirac theory of the gross and fine-structure for one-electron atom takes electron spin and fine structure into account. i.e. it contains effects such as spin-orbit coupling + relativistic effects and Darwin term, which originates from averaging the potential energy over the size of the electron wave.

$$E_D(n; j) = m_e c^2 \,(f(n; j) - 1) \quad [3\text{-}2]$$

$$f(n, j) = \left[1 + \left(\frac{Z\alpha}{n - \varepsilon}\right)^2\right]^{-1/2} \quad [3\text{-}3]$$

$$\varepsilon = j + \frac{1}{2} - \sqrt{(j + \frac{1}{2})^2 - (Z\alpha)^2} \quad [3\text{-}4]$$

E_{RM} : Effects due to finite nuclear mass (relativistic and non relativistic).

E_{QED} QED-Effects (Lamb shift): radioactive corrections to the electron propagator , (Electron self energy, anomalous magnetic moment), vacuum polarization.

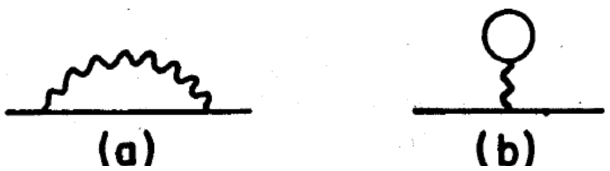


Fig. 3-2: Lowest order QED contributions to the Lamb shift. (a) Electron self energy. (b) Vacuum correction to the potential. The heavy lines represent the electron in an external static nuclear field.

E_{strong} Strong interaction → QED-effects of the vacuum polarization
E_{weak} Weak interaction (via Z-Boson exchange)
E_{HFS} Interaction between magnetic moment of the muon and electron
E_{exotic} possible (non-Standard Model) exotic interaction between electron and „nucleus"

3.2 Hamilton function of the hyperfine interaction

The Hamilton function of an electron in the field of a muon is given by[1]:

$$H = \frac{1}{2m_e}\left[\vec{P} - q\vec{A}_I(\vec{R})\right]^2 + qU_I(\vec{R}) \underbrace{- g_e\mu_B(\frac{\vec{S}}{\hbar})\mathrm{rot}(\vec{A}_I(\vec{R}))}_{-\vec{\mu}_e\vec{B}} \qquad [3\text{-}5]$$

$\vec{R}$ and $\vec{P}$ are position and momentum of the electron, $\vec{S}$ its spin (in units of $\hbar$), q its charge. $\vec{I}$ is the nuclear spin (muon). Let's consider the terms, which originate from the vector potential $\vec{A}$.

$$\vec{A}_I(\vec{R}) = \frac{\mu_0}{4\pi}\frac{\vec{\mu}_\mu \times \vec{R}}{R^3} \qquad [3\text{-}6]$$

Where $\vec{\mu}_\mu$ is the magnetic moment of the muon.

The hyperfine Hamiltonian H_{hf} is obtained, if we retain in [3-5] only the terms linear in $\vec{A}_I$

$$H_{hf} = -\frac{q}{2m_e}\left[\vec{P}\cdot\vec{A}_I + \vec{A}_I\cdot\vec{P}\right] - g_e\mu_B\frac{\vec{S}}{\hbar}\mathrm{rot}(\vec{A}_I(\vec{R})) \qquad [3\text{-}7]$$

and put [3-6] in [3-7].

<u>Coupling of the magnetic moment of the muon with the orbital momentum of the electron</u>

Let's consider the first term in [3-7]. With

$$\vec{L} = \vec{R}\times\vec{P} \qquad [3\text{-}8]$$

and the fact that $\vec{\mu}_\mu$ with $\vec{R}$ and $\vec{P}$ commutes[2], we get:

$$H_{hf}^L = -\frac{\mu_0}{4\pi}\frac{q}{2m_e}2\frac{\vec{\mu}_\mu\cdot\vec{L}}{R^3} = -\frac{\mu_0}{4\pi}2\mu_B\frac{\vec{\mu}_\mu\cdot\frac{\vec{L}}{\hbar}}{R^3} = -\vec{\mu}_\mu\cdot\vec{B}_L \qquad [3\text{-}9]$$

This corresponds to the coupling between the magnetic moment $\vec{\mu}_\mu$ and the magnetic field ($q < 0$)

$$\vec{B}_L = \frac{\mu_0}{4\pi}\frac{q}{m_e}\frac{\vec{L}}{R^3} \qquad [3\text{-}10]$$

---

[1] In this chapter we use [S]=[$\hbar$]

[2] Use: $\vec{P}\cdot\vec{A} \propto \vec{P}\cdot[\vec{\mu}_\mu\times\vec{R}] = \vec{\mu}_\mu\cdot[\vec{R}\times\vec{P}] = \vec{\mu}_\mu\cdot\vec{L}$ and similarly for $\vec{A}\cdot\vec{P}$

This field corresponds to the current generated by the orbiting electron.

(Biot-Savart law: $\vec{B}_L = -\frac{\mu_0}{4\pi} I \int \frac{\vec{R}\times d\vec{R}}{R^3}$, with $I = \frac{-qv}{2\pi R}$)

<u>Coupling with electron spin</u>

Magnetic field created by the muon:

To avoid problems with singularities we consider first a muon with a finite radius ρ_0 and take $R > \rho_0$

With [3-6] and $\vec{B}^{dip} = \mathrm{rot}\vec{A}_I$

$$\vec{B}^{dip}(\vec{R}) = \frac{\mu_0}{4\pi}\left[-\frac{\vec{\mu}_\mu}{R^3} + 3\frac{(\vec{\mu}_\mu \cdot \vec{R})\vec{R})}{R^5}\right] \qquad [3\text{-}11]$$

with $\vec{\mu}_\mu \| \hat{z}$, we get:

$$B_x = \frac{\mu_0}{4\pi} 3\mu_\mu \frac{xz}{R^5}$$

$$B_y = \frac{\mu_0}{4\pi} 3\mu_\mu \frac{yz}{R^5}$$

$$B_z = \frac{\mu_0}{4\pi} \mu_\mu \frac{3z^2 - R^2}{R^5}$$

[3-11] is also valid for R not much larger than ρ_0 , since a spin ½ particle creates a dipolar field.

The magnetic dipole term

If we insert [3-11] in $-g_e\mu_B(\frac{\vec{S}}{\hbar})\mathrm{rot}(\vec{A}_I(\vec{R}))$, we get for the magnetic dipole term (coupling between electron spin and magnetic field, which is generated by the dipole moment of the muon outside its "radius").

$$H_{hf}^{dip} = \frac{\mu_0}{4\pi}\frac{g_e\mu_B}{\hbar}\frac{1}{R^3}\left[\vec{S}\cdot\vec{\mu}_\mu - 3\frac{(\vec{S}\cdot\vec{R})(\vec{\mu}_\mu \cdot \vec{R})}{R^2}\right] = -\vec{\mu}_e \cdot \vec{B}^{dip} \qquad [3\text{-}12]$$

The contact term

It takes into account the contribution of the „internal field " B_i (i.e. of the interaction

between magnetic moment of the muon and electronic spin density at the muon site)
$R \le \rho_0$: The field inside the nucleus (B_i) can be obtained by the explicit integration of the magnetic flux Φ over half a sphere surrounding the dipole, taking into account that the integral is zero.

$$B_i = \frac{\mu_0}{4\pi}\mu_\mu \frac{2}{\rho_0^3} \qquad [3\text{-}13]$$

Contact term in [3-5]:

$$-g_e\mu_B(\frac{\vec{S}}{\hbar})\mathrm{rot}(\vec{A}_I(\vec{R})) = -\vec{\mu}_e \cdot \vec{B}_i$$

The corresponding operator H_{hf}^c is obtained by calculating the matrix elements between the basis wave functions. We get[3]:

$$H_{hf}^c = -\frac{\mu_0}{4\pi}\frac{8\pi}{3}\vec{\mu}_\mu \frac{g_e\mu_B\vec{S}}{\hbar}\delta(\vec{R}) = -\frac{2}{3}\mu_0\vec{\mu}_\mu\vec{\mu}_e\delta(\vec{R}) \qquad [3\text{-}14]$$

Note that the term is finite and does not depend on the choice of ρ_0.

With $H_{hf} = H_{hf}^L + H_{hf}^{dip} + H_{hf}^c$

and $\vec{\mu}_\mu = g_\mu\mu_B^\mu \frac{\vec{I}}{\hbar}$ [3-15]

With the three contributions, the Hamilton operator of the hyperfine interaction ($g_e = 2$) becomes:

$$\boxed{H_{hf} = -\frac{\mu_0}{4\pi}\frac{2\mu_B\mu_B^\mu g_\mu}{\hbar^2}\left[\underbrace{\frac{\vec{I}\cdot\vec{L}}{R^3}}_{\text{e-Angular moment}} + \underbrace{3\frac{(\vec{I}\cdot\vec{R})(\vec{S}\cdot\vec{R})}{R^5} - \frac{\vec{I}\cdot\vec{S}}{R^3}}_{\text{Dipol (e-spin)}} + \underbrace{\frac{8\pi}{3}\vec{I}\cdot\vec{S}\delta(\vec{R})}_{\text{Contact term (e-Spindensity at the muon site)}}\right]} \qquad [3\text{-}16][3]$$

Dipolar and contact fields are also present in the solid. For instance, localized magnetic moments or nuclear moments produce dipolar fields and the spin density of conduction electrons (or delocalized electrons) generates a contact field at the muon site.

[3] (Note here $\mu_B < 0$)

Calculation of the hyperfine structure of the 1s-level

For the 1s level the first two terms of [3-16] are zero, since
$<\vec{L}>$ and $<1s\,|\text{dipolar term}|\,1s> = 0$ because of the spherical symmetry of the 1s-state.
Only the contact term contributes.

Matrix element of the contact term

with $\vec{\mu}_e = g_e \mu_B \frac{\vec{S}}{\hbar}$

$$< n=1, l=0, m_L=0, m_S, m_I \left| -\frac{2}{3}\mu_0 \vec{\mu}_\mu \vec{\mu}_e \delta(\vec{R}) \right| n=1, l=0, m_L=0, m_S, m_I > \qquad [3\text{-}17]$$

$$= A < m_S, m_I \left| \vec{I}\cdot\vec{S} \right| m_S, m_I > \qquad [3\text{-}18]$$

$$A = \frac{2}{3}\mu_0 g_\mu \mu_B^\mu g_e \left|\mu_B\right| \frac{1}{\hbar^2} \underbrace{< n=1, l=0 \left| \delta(\vec{R}) \right| n=1, l=0 >}_{\left|\varphi_{1s}(0)\right|^2} \qquad [3\text{-}19]$$

$$\text{with } < r \,|\, n=1, l=0 > = \frac{1}{\sqrt{\pi a_{Mu}{}^3}} e^{-\frac{r}{a_{Mu}}} = \phi_{1s}(r) \qquad [3\text{-}20]$$

$$< n=1, l=0 \left| \delta(\vec{R}) \right| n=1, l=0 > = \left|\varphi_{1s}(0)\right|^2 = \frac{1}{\pi a_{Mu}{}^3} \qquad [3\text{-}21]$$

$$\boxed{A = \frac{2}{3}\mu_0 g_\mu \mu_B^\mu g_e \left|\mu_B\right| \frac{1}{\pi a_{Mu}^3} \frac{1}{\hbar^2}} \qquad [A] = [\frac{\text{energy}}{\hbar^2}] \qquad [3\text{-}22]$$

With $a_{Mu} = \frac{4\pi\varepsilon_0 \hbar^2}{\bar{m}_{Mu} q^2}$ (Bohr radius), $\bar{m}_{Mu} = \frac{m_e m_\mu}{m_e + m_\mu} = \frac{m_e}{1 + \frac{m_e}{m_\mu}}$ (takes into account the finite “nuclear” mass), $\alpha = \frac{q^2}{4\pi\varepsilon_0 \hbar c}$ (fine structure constant in SI units) and $\varepsilon_0 \mu_0 = \frac{1}{c^2}$

For spectroscopy [3-22] can also be written in terms of precisely known quantities:

$$A = \frac{2}{3} g_\mu g_e \frac{m_e}{m_\mu} m_e c^2 \alpha^4 (1 + \frac{m_e}{m_\mu})^{-3} \frac{1}{\hbar^2} \qquad [3\text{-}23]$$

Summarizing the contact operator for the 1s state can be simplified to

$$\boxed{\begin{array}{l} H_{hf}^{c} = A\vec{I}\cdot\vec{S} \\ \\ \text{With } A>0 \text{ and } [I]=[S]=[\hbar] \end{array}} \qquad [3\text{-}24]$$

Eigenvalues and eigenstates of the contact term of the 1s-level

The degeneracy of the 1s-level is 4-fold. Instead of the basis

$$| \; S = \frac{1}{2}, I = \frac{1}{2}, m_S, m_I > \qquad [3\text{-}25]$$

we take the basis

$$| \; F, m_F > \qquad [3\text{-}26]$$

F is the total moment eigenvalue of the operator $\vec{F} = \vec{S} + \vec{I}$: [3-27]

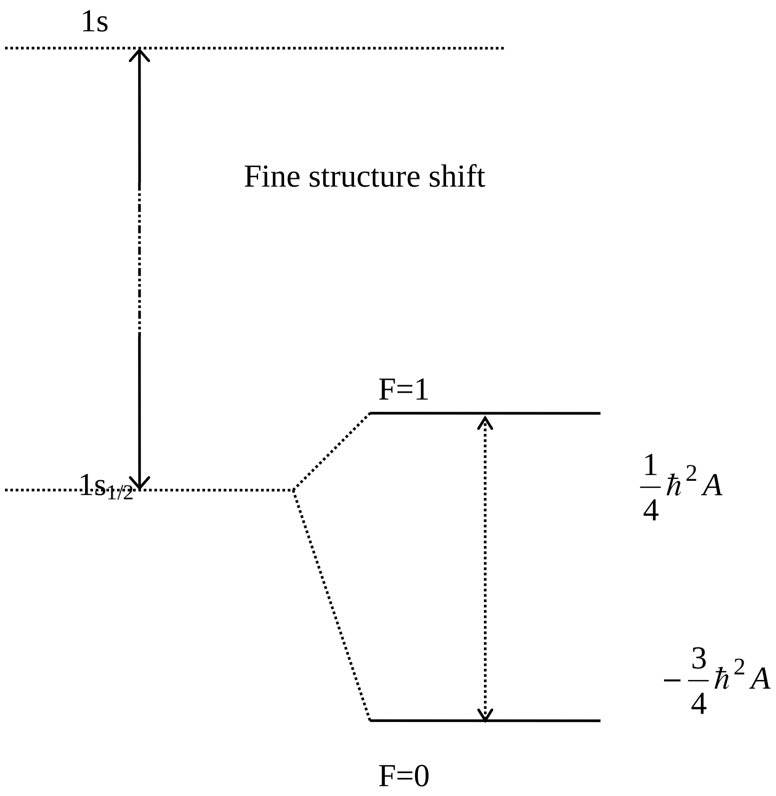
$\vec{I} \cdot \vec{S}$ is diagonal in the basis [3-26]. With L=0 (1s level)

$$A\vec{I}\cdot\vec{S} = \frac{A}{2}(F^2 - I^2 - S^2) \qquad [3\text{-}28]$$

$$A\vec{I}\cdot\vec{S}\left|F, m_F\right> = A\frac{\hbar^2}{2}\left[F(F+1) - I(I+1) - S(S+1)\right]\left|F, m_F\right> \qquad [3\text{-}29]$$

$$= A\frac{\hbar^2}{4} \qquad F = 1$$

$$= -A\frac{3\hbar^2}{4} \qquad F = 0$$



Eq. [3-23] is not accurate enough for high precision spectroscopy of muonium or of hydrogen and positronium. One has to consider additional correction terms arising from QED, weak interaction, and eventually exotic interactions.

However, the hyperfine-Hamilton function has still the form:

$$H_{hf}^{c} = A\vec{I}\cdot\vec{S} \qquad [3\text{-}30]$$

If we write the splitting in terms of frequency we get:

$$E_{hfs}(F=1) - E_{hfs}(F=0) = h\Delta\nu_{hfs} = \hbar^2 A \qquad [3.31]$$

$$\Delta\nu_{hfs}^{Mu} = \frac{16}{3}(Z\alpha)^2 \frac{R_y}{h}\frac{\mu_\mu}{\mu_B}\left[1+\frac{m_e}{m_\mu}\right]^{-3}\left(1+\varepsilon_{rad}+\varepsilon_{rec}+\varepsilon_{rad-rec}\right)+\Delta\nu_{weak}+\Delta\nu_{exotic} \qquad [3\text{-}32]$$

where $R_y = \dfrac{\alpha^2 m_e c^2}{2}$ and Z=1

Theoretical value:

$$\Delta\nu_{hfs}^{th} == 4\,463\,302\,891\,(272)\ \text{Hz}\ (63\ \text{ppb}) \qquad [3\text{-}33]$$

Better known for muonium than for H. For H one has to consider additional terms due to the proton structure:

$+\ \varepsilon_{\text{nuclear radius}} +\ \varepsilon_{\text{nuclear polarization}}$.

The theoretical uncertainty is 560 ppb (whereas the experimental uncertainty is presently 0.6 ppt).

3.3 Spectroscopy of the hyperfine splitting in Mu

Method: Microwave spectroscopy of the ground state in an external magnetic field.

We consider the Zeeman effect on the 1s level.

Hamilton function:

$$H = -\vec{\mu}_\mu \cdot \vec{B} - \vec{\mu}_e \cdot \vec{B} + A\vec{I}\cdot\vec{S}$$

with $\vec{B} \parallel \hat{z}$ and $\vec{\mu}_\mu = \gamma_\mu \vec{I}$ $\quad$ ($[I]=\hbar$) [3-34]

and using the Larmor frequencies of muon and electron:

$$H = \omega_\mu I_z + \omega_e S_z + A\vec{I}\cdot\vec{S} \qquad [3\text{-}35]$$

where:

$$\omega_\mu = -\frac{g_\mu}{2}\frac{|e|}{m_\mu}B \quad , \quad \omega_\mu < 0$$

$$\omega_e = \frac{g_e}{2}\frac{|e|}{m_e}B \quad , \quad \omega_e > 0 \qquad [3\text{-}36]$$

To determine the energy eigenvalues, we must diagonalize the matrix of the Hamilton function *H* [3-35].

We obtain as energy eigenvalues:

$$E_1 = \frac{A\hbar^2}{4} + \frac{\hbar}{2}(\omega_e + \omega_\mu) = \frac{A\hbar^2}{4} + \frac{1}{2}(g_e|\mu_B| - g_\mu \mu_B^\mu)B$$

$$E_2 = \frac{A\hbar^2}{4} - \frac{\hbar}{2}(\omega_e + \omega_\mu) = \frac{A\hbar^2}{4} - \frac{1}{2}(g_e|\mu_B| - g_\mu \mu_B^\mu)B$$

$$E_3 = -\frac{A\hbar^2}{4} + \sqrt{(\frac{A\hbar^2}{2})^2 + \frac{\hbar^2}{4}(\omega_e - \omega_\mu)^2} = -\frac{A\hbar^2}{4} + \frac{A\hbar^2}{2}\sqrt{1+x^2} \qquad [3\text{-}37]$$

$$E_4 = -\frac{A\hbar^2}{4} - \sqrt{(\frac{A\hbar^2}{2})^2 + \frac{\hbar^2}{4}(\omega_e - \omega_\mu)^2} = -\frac{A\hbar^2}{4} - \frac{A\hbar^2}{2}\sqrt{1+x^2}$$

Breit-Rabi diagram

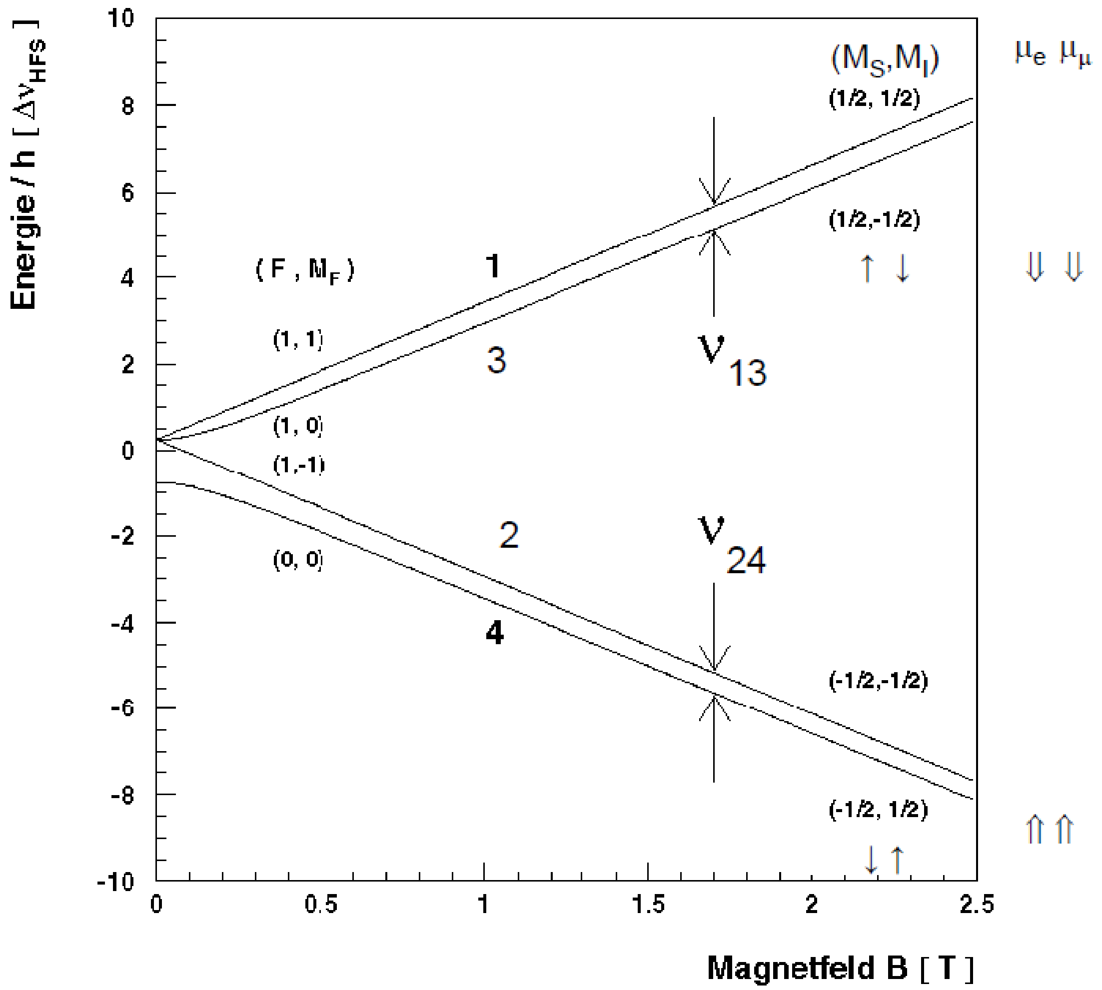


Fig. 3-3: Energy-level diagram for muonium in the $1^2S_{1/2}$ ground state in a magnetic field. At zero magnetic field the energy difference between the F=1 and F=0 states is the hyperfine splitting $h\Delta\nu_{hfs}$.

The field is generally expressed in terms of the dimensionless parameter

$$x = \frac{(g_e|\mu_B| + g_\mu \mu_B^\mu)B}{A\hbar^2} = \frac{(g_e|\mu_B| + g_\mu \mu_B^\mu)B}{h\Delta\nu_{hfs}} \equiv \frac{B}{B_0} \qquad [3\text{-}38]$$

B_0 is the field where the Zeeman splitting of the electron and the muon is equal to the hyperfine splitting. For muonium B_0=0.158 T.

In analogy we obtain the eigenstates:

$$\begin{aligned}
&|\ 1>=|\ \mathrm{M_S}=\frac{1}{2},\mathrm{M_I}=\frac{1}{2}>\ =|\ \mathrm{F}=1,\mathrm{M_F}=1>\\
&|\ 2>=|\ \mathrm{M_S}=-\frac{1}{2},\mathrm{M_I}=-\frac{1}{2}>\ =|\ \mathrm{F}=1,\mathrm{M_F}=-1>\\
&|\ 3>=\sin\beta|\ \mathrm{M_S}=-\frac{1}{2},\mathrm{M_I}=\frac{1}{2}>+\cos\beta|\ \mathrm{M_S}=+\frac{1}{2},\mathrm{M_I}=-\frac{1}{2}>\\
&|\ 4>=\cos\beta|\ \mathrm{M_S}=-\frac{1}{2},\mathrm{M_I}=\frac{1}{2}>-\sin\beta|\ \mathrm{M_S}=+\frac{1}{2},\mathrm{M_I}=-\frac{1}{2}>
\end{aligned} \tag{3-39}$$

where

$$\cos\beta=\frac{1}{\sqrt{2}}\left[1+\frac{x}{(1+x^2)^{1/2}}\right]^{1/2}$$

$$\sin\beta=\frac{1}{\sqrt{2}}\left[1-\frac{x}{(1+x^2)^{1/2}}\right]^{1/2}$$

We express the energy differences as frequencies:

$$E_1 - E_3 = h\nu_{13}$$
$$E_2 - E_4 = h\nu_{24} \qquad [3\text{-}40]$$

Then we get for the sum or difference of the transition energies:

$$h\nu_{13} + h\nu_{24} = h\Delta\nu_{hfs}^{Mu} = \hbar^2 A$$
$$h\nu_{13} - h\nu_{24} = 2\mu_B^{\mu} g'_{\mu} B + h\Delta\nu_{hfs}^{Mu}\left[(1+x^2)^{1/2} - x\right] \qquad [3\text{-}41]$$

$$x = \frac{(g'_e|\mu_B| + g'_\mu \mu_B^\mu)B}{h\Delta\nu_{hfs}^M}$$

To take into account the relativistic binding corrections in muonium we use in [3-41] g'_e and g'_μ instead of the values for free particles g_e and g_μ.

$$g'_\mu = g_\mu[1 - \frac{\alpha^2}{3} + \frac{\alpha^2}{2}\frac{m_e}{m_\mu}]$$
$$g'_e = g_e[1 - \frac{\alpha^2}{3} + \frac{\alpha^2}{2}\frac{m_e}{m_\mu} + \frac{\alpha^3}{4\pi}] \qquad [3\text{-}42]$$

B can be expressed as a function of μ_p and of the NMR frequency.

$$h\nu_p = 2\mu_p B \qquad [3\text{-}43]$$

From the sum of the transition frequencies $\Delta\nu_{hfs}^{Mu}$ is determined and from the difference (eq. [3-41]) by using $\mu_\mu = \frac{g_\mu \mu_B^\mu}{2}$ we obtain also the ratio of the muon and proton magnetic moments $\frac{\mu_\mu}{\mu_p}$.

Principle of the experiment (W. Lin et al., Phys. Rev. Lett., **82** (1999) 711)

- Stop polarized positive muons in Kr in a magnetic field $\vec{B}$ antiparallel to $\vec{I}$ (initial muon spin direction)
- In Kr muonium is formed in the states (each with 50% probability since the electrons are unpolarized)

  $|M_S = -\frac{1}{2}, M_I = -\frac{1}{2}>$ (level 2)

  and $|M_S = +\frac{1}{2}, M_I = -\frac{1}{2}>$ (level 3)
- With microwaves one induces the transitions in level 4 ($\rightarrow h\nu_{24}$) and in level 1 ($\rightarrow h\nu_{13}$) .
- The transition frequencies are determined from the positrons rates with and without microwave field (one can vary either the microwave frequency or the magnetic field)

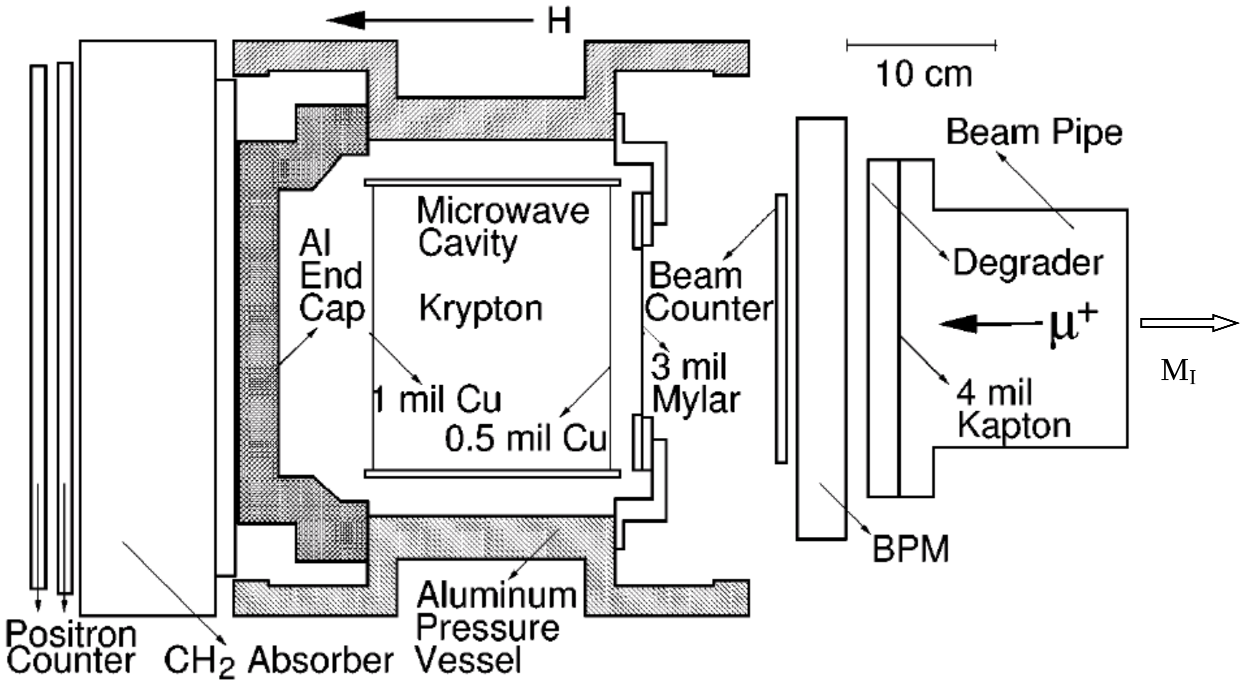


Fig. 3-4: A schematic view of the experimental apparatus

Positron signal (as a function of the magnetic field or of the microwave frequency)

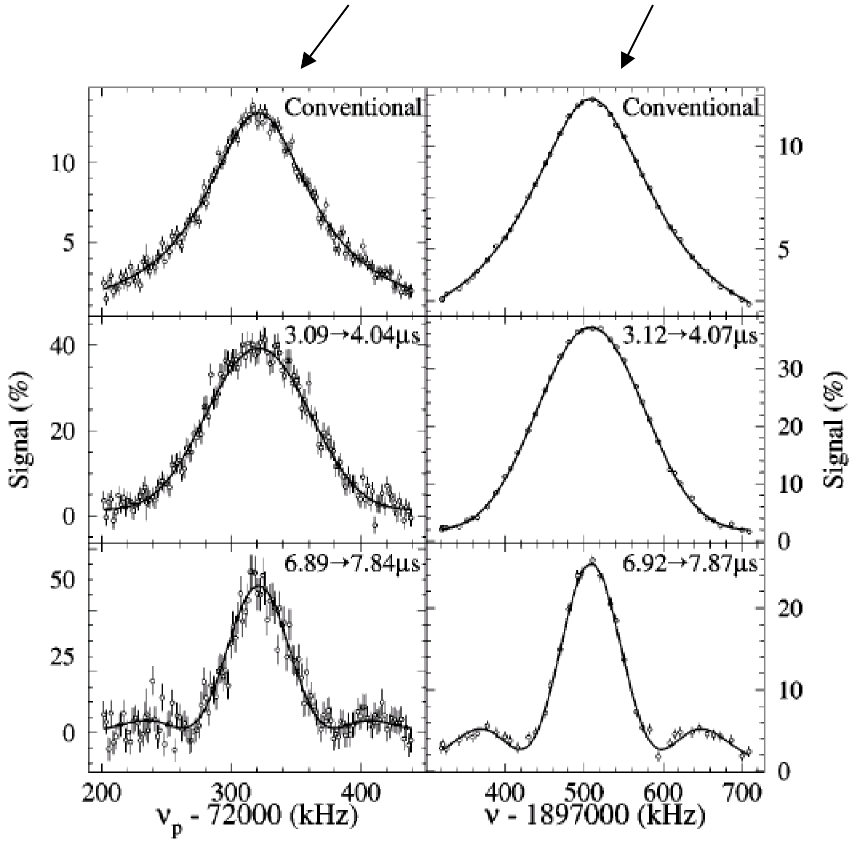


Fig. 3-5: Resonance curves obtained by sweeping the magnetic field using a conventional method, and from different time windows after muonium production are shown on the left. Microwave frequency sweep curves are on the right. The solid curves are fits to the theoretical line shape |

Result:

$\Delta\nu_{hfs}^{exp}$ = 4 463 302 765(53) Hz (12 ppb) [3-44]

$\Delta\nu_{hfs}^{th}$ = 4 463 302 891 (272) Hz (63 ppb)

μ_μ/μ_p = 3.183 345 13(39) (122 ppb) [3-45]

(Ref. Liu et al., Phys. Rev. Lett. **82** 711 (1999))

From [3-45] via $\frac{m_\mu}{m_e} = \frac{g_\mu}{2}\frac{\mu_p}{\mu_\mu}\frac{\mu_B^e}{\mu_p}$:

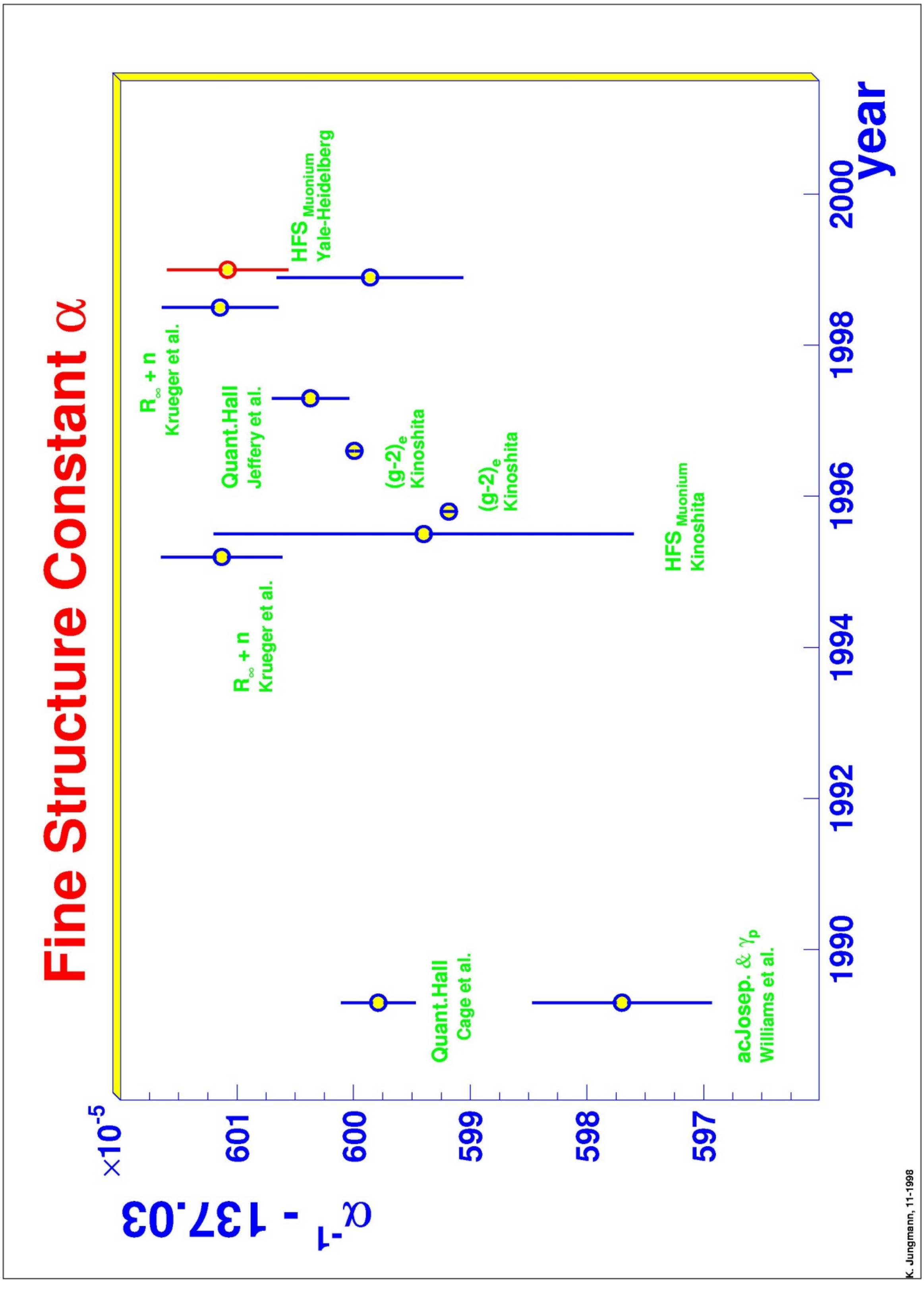
m_μ/m_e = 206.768 277(24) (120 ppb) can be determined [3-46]

Alternatively one can use m_μ/m_e or α as parameter in eq. [3-32] and determine them from the experimental result for $\Delta\nu_{hfs}^{exp}$.

For instance with m_μ/m_e from [3-46] one gets:

α^{-1} = 137.0359963(80) (58 ppb)

Fine structure constant: summary of results



3.4 Measurement of the 1s-2s transition in muonium

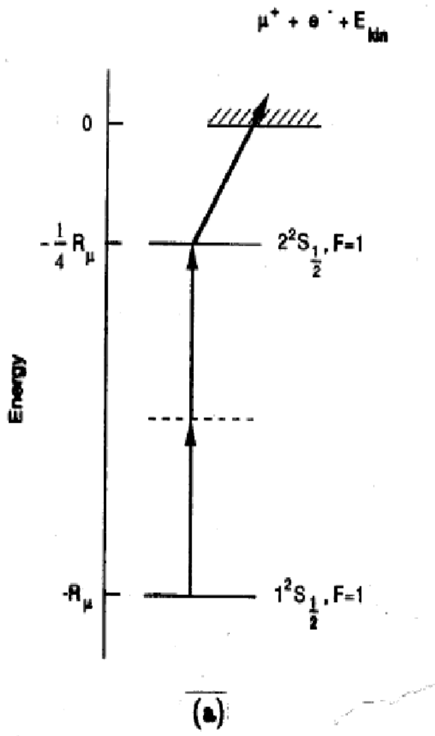
(Doppler free 2-photons spectroscopy)

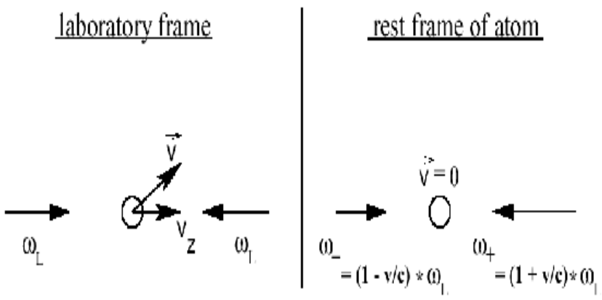
- 1 photon transition not allowed (because of $\Delta l = \pm 1$)
- Gross structure interval

  $\nu_{1s2s} = \frac{3}{4} c R_\infty (1 - \frac{m_e}{m_\mu}) = 2.45 \cdot 10^{15}\,\mathrm{Hz}$ $\qquad hcR_\infty \equiv R_y = m_e c^2 \frac{\alpha^2}{2}$ [3-47]

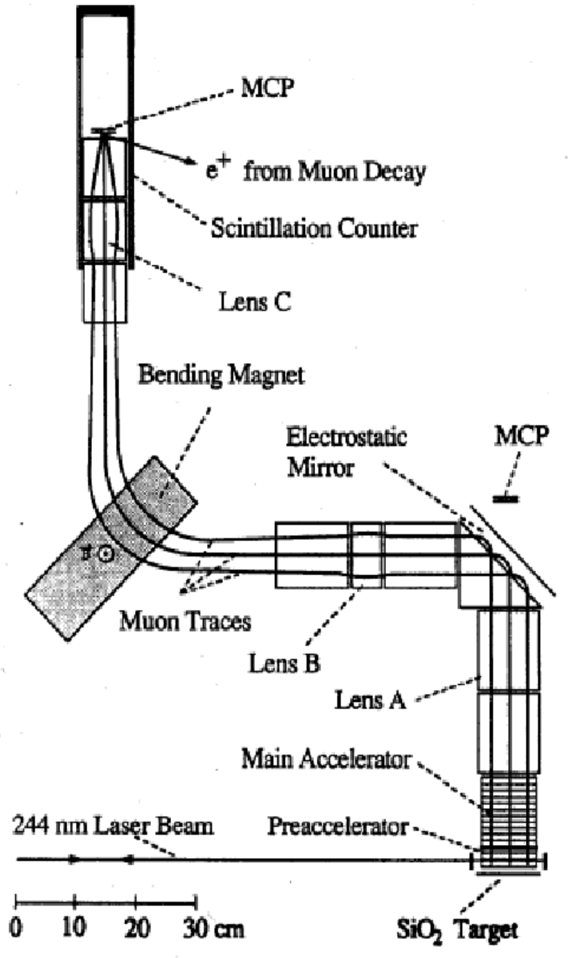
  R_∞ (Rydberg constant) is known to 8. 10^{-12}
- Natural width due to lifetime of the muon: ν_μ = 145 kHz → $\frac{\nu_\mu}{\nu_{1s2s}} \approx 6 \cdot 10^{-11}$
- A measurement of ν_{1s2s} at the 10^{-9} level allows an accuracy of $\frac{m_\mu}{m_e}$ given by $10^{-9} \frac{m_\mu}{m_e} \approx 10^{-7}$

Measurement principle (Doppler free spectroscopy)





Principle of the 1s-2s muonium experiment. a) The transition between the 1s- and 2s- levels is induced by the absorption of two counterpropagating photons (λ= 244.2 nm). The metastable 2s-state is ionized by a third photon. b) The transition via two photon absorption is to first order Doppler free.



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Apparatus for the 1s-2s experiment at the Rutherford Appleton Laboratory (pulsed muon source, UK).

Production of thermal muonium in vacuum. Efficiency for different materials. The beam momentum in optimized for maximum efficiency.

Target material	*Target density [mg/cm³]*	*Target Thickness [mg/cm²]*	**Optimum Muonium Fraction μ^+e^-/μ_{stop} [%]**
SiO_2 powder	32	4.6	17(1)
SiO_2 powder	32	2.8	15.9(3.6)
SiO_2 powder	32	9.0	8.27(31)
SiO_2 aerogel	5	7.5	2.32(13)
SiO_2 aerogel	18	9	1.57(20)
W Foil (2130K)	19.3	96.5	4(2)
C_{60}/C_{70} Fullerenes	≈1400	≈210	1.85(23)
Cotton	10	3.6	2.25(16)
Cotton coated with SiO2 powder	17	5.8	11.43(31)
Microchannel Plate	≈2000	≈100	2.44(31)

Results

$\Delta\nu_{1s2s}^{Exp} = 2\,455\,528\,941.0\,(9.8)\ \mathrm{MHz}$ (4 ppb)

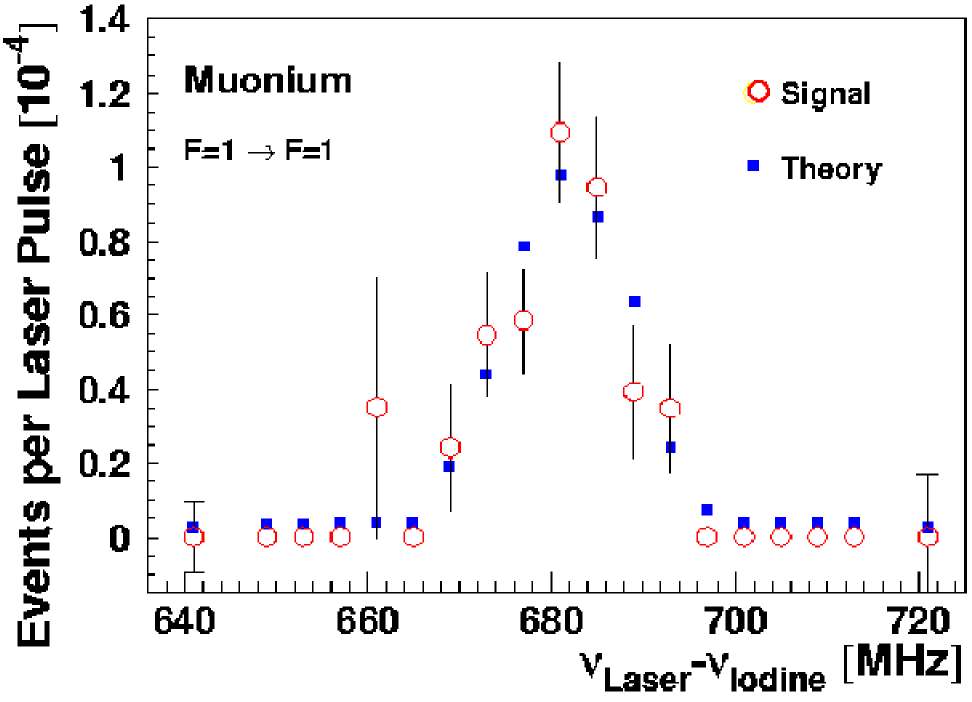
From the comparison with the theoretical value $\Delta\nu_{1s2s}^{Theor} = 2\,455\,528\,934.5\,(3.6)\ \mathrm{MHz}$

→ $\frac{m_\mu}{m_e} = 206.76838(16)$ (0.77 ppm)

or alternatively: from the comparison with the theory and the fact that the dominant term in [3-47] is proportional to $R_y \propto \alpha^2 \propto q_\mu{}^2 \cdot q_e{}^2 \propto (\frac{q_\mu}{q_e})^2 (\frac{q_e{}^4}{\alpha_{true}{}^2}) \propto (\frac{q_\mu}{q_e})^2$

→ $(\frac{q_\mu}{q_e}) = -1 - 1.0(2.0) \cdot 10^{-9}$ (2.0 ppb)

This is a test of charge equality between two particle generations.



Muonium
F=1 → F=1
Signal
Theory
Events per Laser Pulse [10^-4]
1.4
1.2
1
0.8
0.6
0.4
0.2
0
640
660
680
700
720
ν_Laser-ν_Iodine [MHz]

Muon mass results, summary:

Extracted from experiments:

$$\frac{m_{\mu^+}}{m_e} \; (\mu SR\ Br) = 206.768\,35\,(11) \quad (0.53\ ppm)$$

$$\frac{m_{\mu^+}}{m_e} \; (\mu^-\ Atoms) = 206.768\,30\,(64) \quad (3.1\ ppm)$$

$$\frac{m_{\mu^+}}{m_e} \; (M\ 1s\text{-}2s) = 206.768\,38\,(16) \quad (0.77\ ppm)$$

$$\frac{m_{\mu^+}}{m_e} \; (\mu_\mu) = 206.768\,270\,(24) \quad (0.12\ ppm)$$

Using the Muonium hyperfine structure measurement and the theory:

$$\frac{m_{\mu^+}}{m_e} \; (M_{hfs}) = 206.768\,267\,0\,(55) \quad (0.027\ ppm)$$

Value in Particle Data Book (2014):

$$\frac{m_{\mu^+}}{m_e} = 206.768\,284\,3(52) \quad (0.025\ ppm)$$

Summary of precision experiments: