Stochastic programming formulations of coherent multiperiod risk measurement

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Vilnius, 9. July 2012

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Single-period risk-adjusted values, CVaR

- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- Financial value $X \in L^\infty$

Representation of coherent risk-adjusted values [Artzner et al., 2007]:

$$\pi[X] = \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[X],$$

with (convex) set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$.

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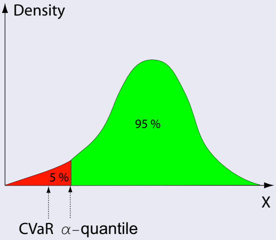
Special case: Conditional-Value-at-Risk

$\alpha \in (0,1)$; finitely discrete $\mathcal{F}$, realizations $x_1, \ldots, x_n$, probabilities $p_1, \ldots, p_n > 0$:

$$\mathrm{CVaR}^\alpha[X] := \min_{\left\{0 \le \frac{q_i}{p_i} \le \frac{1}{\alpha} \,\middle|\, \sum_{i=1}^n q_i = 1,\ q_i \ge 0\right\}} \sum\nolimits_{i=1}^n q_i x_i .$$



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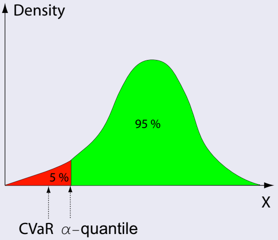
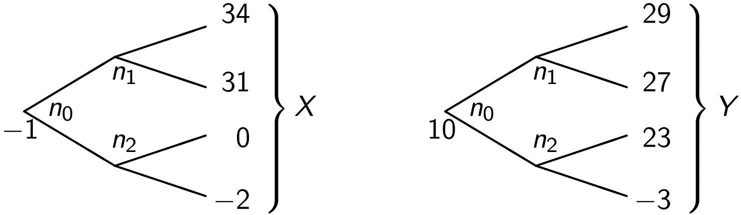
Generally: $$\mathrm{CVaR}^\alpha[X] := \min_{\left\{\mathbb{Q} \,\middle|\, \frac{d\mathbb{Q}}{d\mathbb{P}} \le \frac{1}{\alpha}\right\}} \mathbb{E}_{\mathbb{Q}}[X]$$



Time-inconsistency of CVaR (for final values)

- Binary scenario tree with random values X and Y on terminal nodes
- Terminal nodes are equiprobable
- $\alpha = 1/2$; $\mathrm{CVaR}^{\alpha}[\cdot]$ conditional on non-terminal nodes n_0, n_1, n_2:



There is only a single tree (duplicated for readability).

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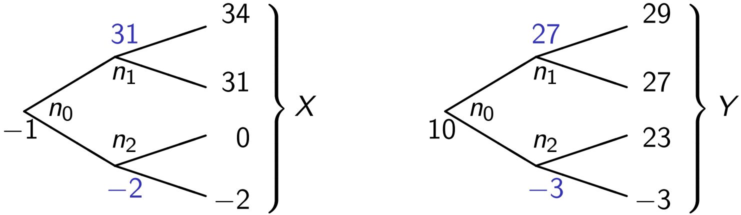

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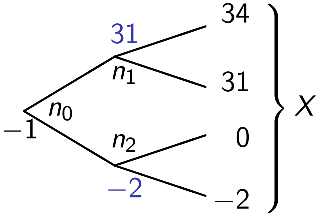

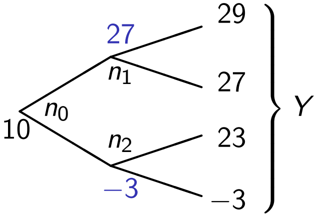

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Today (i.e. n_0), Y has larger risk-adjusted value; tomorrow (n_1 and n_2), X has larger risk-adjusted value. → mean-risk criterium is no cure

Monte-Carlo analysis of time-consistency

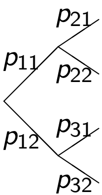
- Tree with 2 time steps (as before)
- Equiprobable terminal nodes (as before)
- X and Y values randomly sampled from $\mathrm{U}[0,1]$
- Counts of time consistency (for samples yielding an order over time):

branches (at non-terminal nodes)	α	X,Y time-consistent w.r.t. CVaR
2	0.5	95%
2	0.4	98%
2	0.3	99%
4	0.2	99%
4	0.1	99%
8	all	$\sim$ 100%

Multiperiod extension of CVaR: stepwise-CVaR

- Time steps: $t = 0, 1, \ldots, T$
- Finite setting: Gain of information given by scenario tree
- Identification: probability measure $\mathbb{P}, \mathbb{Q}, \ldots \;\widehat{=}$ transition probabilities on the tree

Example: $\mathbb{P} \;\widehat{=}\; (p_{11}, p_{12}, p_{21}, p_{22}, p_{23}, p_{24})$



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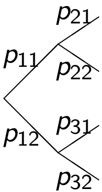

Def.: <u>Stepwise-CVaR set</u> $\mathcal{P}^\alpha$ of probability measures:

$$\mathcal{P}^\alpha := \left\{ (q_{11}, q_{12}, q_{21}, q_{22}, q_{31}, q_{32}) \;\middle|\; \frac{q_{ij}}{p_{ij}} \le \frac{1}{\alpha}, \quad \sum_{j=1}^{2} q_{ij} = 1,\; q_{ij} \ge 0 \quad \forall i, j \right\}$$

Risk-adjusted value for stochastic processes

- Notation: scenario tree $\widehat{=}$ finite filtration $(\mathcal{F}_t)_{t=0,\dots,T}$
- Stepwise-CVaR set $\mathcal{P}^\alpha$
- $(X) := (X_t)_{t=0,\dots,T}$ stochastic process of values, X_t $\mathcal{F}_t$-measurable $\forall t$

Risk-adjusted value process [Artzner et al., 2007]:

$$R_t^{(X_0,\dots,X_T)} = R_t^{(X)} := \begin{cases} X_T, & \text{if } t = T, \\ \min\Big(X_t, \inf_{\mathbb{Q}\in\mathcal{P}^\alpha} \mathbb{E}_{\mathbb{Q}}[R_{t+1}^{(X)}|\mathcal{F}_t]\Big), & \text{if } t = 0,\dots,T-1. \end{cases}$$

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- $R_0^{(X)} \in \mathbb{R}$: risk-adjusted value measured at time $t = 0$
- $(R_t^{(\cdot)})_{t=0,\dots,T}$ is coherent, and time-consistent (arbitrary $\mathcal{P}$ possible)

$\rightarrow$ Def. on next slide

Time-consistency, Coherency

Time-consistency preserves orderings over time:

$$R_t^{(X_1,\dots,X_T)} \geq R_t^{(X_1,\dots,X_{t-1},Y_t,\dots,Y_T)} \text{ a.s.} \Rightarrow R_0^{(X_1,\dots,X_T)} \geq R_0^{(X_1,\dots,X_{t-1},Y_t,\dots,Y_T)}.$$

Coherency (multiperiod):

Example for $R_0^{(\cdot)}$, $(X+Y) := (X_t + Y_t)_{t=0,\dots,T}$:

(i) $R_0^{(X+Y)} \geq R_0^{(X)} + R_0^{(Y)}, \quad \forall\, (X),(Y)$, adapted,

(ii) $R_0^{(\lambda X)} = \lambda R_0^{(X)}, \quad \forall \lambda \geq 0,\ \lambda \in \mathbb{R}$,

(iii) $X_0 \leq Y_0, \dots, X_T \leq Y_T$ a.s. $\implies R_0^{(X)} \leq R_0^{(Y)}$,

(iv) $R_0^{(X+c)} = R_0^{(X)} + c \quad \forall c \in \mathbb{R}$.

Approach on product space of state and time

1. Previous slides: Recursive calculation $\rightarrow$ Coherency, Time Consistency
2. Next: Extend CVaR formally to product space $\rightarrow$ Coherency

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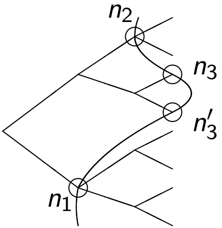
Definition

Stopping time $\tau : \Omega \to \{0, 1, \ldots, T\}$, with $\{\tau = t\} \in \mathcal{F}_t \ \forall t$.
$\mathcal{T}$: set of stopping times.

Example:

$\tau : \Omega \to \{0, 1, 2, 3\}$,
$\Omega := n_1 \cap n_2 \cap n_3 \cap n_3'$



$\tau|_{n_2} = 2,$
$\tau|_{n_3} = 3,$
$\tau|_{n_3'} = 3,$
$\tau|_{n_1} = 1.$

$X_\tau(\omega) := X_{\tau(\omega)}(\omega)$, for a process $(X_t)_{t=0,\ldots,T}$

Submitted

Conclusion

- Time-consistent and a formal coherent extension of CVaR can be incorporated in multi-stage mean-risk problems
- Which properties of risk measurement are important in practice?

Selected References:

- **Artzner et al.:** Coherent Multiperiod Risk-Adjusted Values And Bellman's Principle (2007)
- **Eichhorn, Römisch:** Polyhedral Risk Measures in Stochastic Programming (2005)
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