The partonic structure of the electron at the next-to-leading logarithmic accuracy in QED

based on 1911.12040, 2020.xxxxx
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Outline

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QED Parton Distribution Functions

Analytical solutions

Results in the $\overline{\text{MS}}$ scheme

Change of subtraction scheme

Summary

Motivation

High-priority future initiatives

A. An electron-positron Higgs factory is the highest-priority next collider. For the longer term, the European particle physics community has the ambition to operate a proton-proton collider at the highest achievable energy. Accomplishing these compelling goals will require innovation and cutting-edge technology:

• the particle physics community should ramp up its R&D effort focused on advanced accelerator technologies, in particular that for high-field superconducting magnets, including high-temperature superconductors;

• Europe, together with its international partners, should investigate the technical and financial feasibility of a future hadron collider at CERN with a centre-of-mass energy of at least 100 TeV and with an electron-positron Higgs and electroweak factory as a possible first stage. Such a feasibility study of the colliders and related infrastructure should be established as a global endeavour and be completed on the timescale of the next Strategy update.

The timely realisation of the electron-positron International Linear Collider (ILC) in Japan would be compatible with this strategy and, in that case, the European particle physics community would wish to collaborate.

[2020 UPDATE OF THE EUROPEAN STRATEGY FOR PARTICLE PHYSICS]

Not unreasonable to assume that the future of high-energy physics will involve an e^+e^- collider.

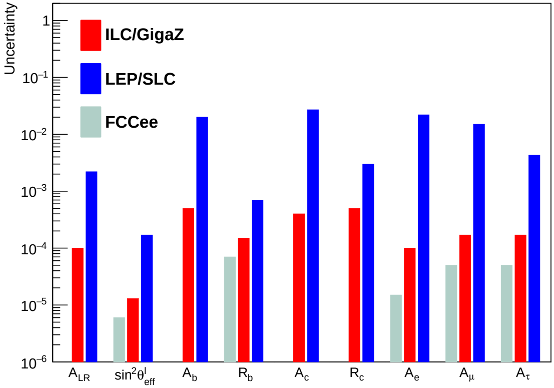
From the theoretical point of view, whether such a collider will be a linear or a circular one is relatively unimportant.

Motivation

Very clean environment favours more precise measurements.

Expected 0.01% relative uncertainty on experimental measurements at ILC, at FCCee possibly even smaller.

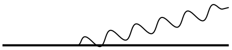
Is possible to have theory under control at that level?



[1908.11299]

PDFs in e^+e^- collisions

Presence in the cross section of potentially large logarithms, due to colliner photon emissions in the initial state:



$$\left| \quad \cdots \quad \right|^2 \propto \alpha \log\left(\frac{s}{m_e^2}\right)$$

They are process-independent and can be resummed by means of the PDF formalism:

1. Collect this logarithm in a PDF $\Gamma(z, \mu_0^2)$ at an initial scale $\mu_0^2 \simeq m_e^2$ (*initial condition*);
2. evolve until $\mu^2 \simeq s$ with the DGLAP equations:

$$\frac{\partial \Gamma(z, \mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} [\mathbb{P} \otimes \Gamma](z, \mu^2),$$

which resums the $\log(\mu^2/\mu_0^2)$ terms to all orders.

QED factorisation formula

Physical *particle*-level cross section $d\sigma_{e^+e^-}$ as a convolution of a **massless** *parton*-level cross section $d\hat{\sigma}_{ij}$ with QED PDFs Γ_{i/e^+} and Γ_{j/e^-}, with $i, j = e^+, e^-, \gamma$:

$$d\sigma_{e^+e^-}(p_{e^+}, p_{e^-}) = \sum_{ij=e^\pm,\gamma} \int dz_+ dz_- \, \Gamma_{i/e^+}(z_+, \mu^2, m^2) \, \Gamma_{j/e^-}(z_-, \mu^2, m^2) \times d\hat{\sigma}_{ij}(z_+ p_{e^+}, z_- p_{e^-}, \mu^2) + \mathcal{O}\left(\frac{m}{E}\right)$$

PDFs Γ_{i/e^+} and Γ_{j/e^-} and massless cross section $d\hat{\sigma}_{ij}$ depend on the subtraction scheme, physical cross section $d\sigma_{e^+e^-}$ is independent of it (order by order). We work in the $\overline{\text{MS}}$ scheme.

N.B. We focus on $\Gamma_{i/e^-} \equiv \Gamma_i$, i.e. the PDFs of the incoming electron.
i.e. we derive the electron/positron/photon PDF of an unpolarised electron.

Towards NLL resummation

- QED PDFs currently known at leading-logarithmic (LL) accuracy i.e. resummation of $(\alpha \log(E/m_e))^k$ terms. Initial condition: just a delta function

$$\Gamma_i^{[0]}(z, \mu_0^2) = \delta_{ie^-}\delta(1-z)$$

- Goal: **reach NLL accuracy** i.e. extend the resummation to the $\alpha(\alpha \log(E/m_e))^k$ terms. Initial conditions given in [Frixione; 1909.03886]:

$$\Gamma_{e^-}^{[1]}(z, \mu_0^2) = \left[\frac{1+z^2}{1-z}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-z) - 1\right)\right]_+ + K_{ee}(z)(= 0 \text{ in } \overline{\text{MS}})$$

$$\Gamma_{\gamma}^{[1]}(z, \mu_0^2) = \frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right) + K_{\gamma e}(z)(= 0 \text{ in } \overline{\text{MS}})$$

$$\Gamma_{e^+}^{[1]}(z, \mu_0^2) = 0$$

Solutions of evolution equations

[Bertone, Cacciari, Frixione, GS; 1911.12040]

$$\frac{\partial \Gamma(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi}[\mathbb{P} \otimes \Gamma](z,\mu^2)$$

- Numerical: public code `github.com/gstagnit/ePDF`
  - Evolution equation solved in Mellin space, by means of a discretised path-ordered product (see e.g. [Bonvini; 1212.0480]) or U-matrix formalism (see e.g. [Vogt; hep-ph/0408244])
  - Numerical inverse Mellin transform with an algorithm based on an optimized path in the complex plane (Talbot path)
- Analytical
  - recursive solution (calculated up to $\mathcal{O}(\alpha^3)$), valid in the small and intermediate z-range
  - asymptotic large-z solution (to all order in α)
  - additive matching to obtain predictions in the whole z range

Crucial difference w.r.t. hadronic PDFs:
QED PDFs are entirely calculable with perturbative techniques.

En passant

By introducing the non-singlet and singlet (with photon) combinations:

$$\Gamma_{\text{NS}} = \frac{1}{2}\left(\Gamma_{e^-} - \Gamma_{e^+}\right), \quad \Gamma_{\text{S}} = \frac{1}{2}\left(\Gamma_{e^+} + \Gamma_{e^-}\right), \quad \Gamma_{\gamma},$$

non-singlet component decouples from the singlet-photon system:

$$\frac{\partial \Gamma_{\text{NS}}}{\partial \log \mu^2} = \frac{\alpha}{2\pi} P_{\text{NS}} \otimes \Gamma_{\text{NS}} .$$

$$\frac{\partial}{\partial \log \mu^2} \begin{pmatrix} \Gamma_{\text{S}} \\ \Gamma_{\gamma} \end{pmatrix} = \frac{\alpha}{2\pi} \mathbb{P}_{\text{S}} \otimes \begin{pmatrix} \Gamma_{\text{S}} \\ \Gamma_{\gamma} \end{pmatrix} ,$$

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Analytic recursive solutions

- Solving the evolution equations order by order in perturbation theory. First make an ansatz, and then find a recurrence relation.
- Approach already known at LL, see e.g. [Jadach, Skrzypek; (1991)], [Cacciari et al.; (1992)]. We extended it at **NLL with α running**.
- Result written as a power series in the t variable:

$$\Gamma(z,\mu^2) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(J_k^{\text{LL}}(z) + \frac{\alpha(t)}{2\pi} J_k^{\text{NLL}}(z) \right)$$

with:

$$t = \frac{1}{2\pi b_0} \log \frac{\alpha(\mu)}{\alpha(\mu_0)} = \frac{\alpha(\mu)}{2\pi} \log \frac{\mu^2}{\mu_0^2} + \mathcal{O}(\alpha^2)$$

i.e. LL terms: t^k and NLL terms: $(\alpha/(2\pi))\, t^k$.

Recurrence relations at NLL

$$\mathcal{J}_k^{\mathrm{LL}} = \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1}^{\mathrm{LL}}$$

$$\mathcal{J}_k^{\mathrm{NLL}} = (-)^k (2\pi b_0)^k \mathcal{J}_0^{\mathrm{NLL}} + \sum_{p=0}^{k-1} (-)^p (2\pi b_0)^p < \left(\mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{NLL}} + \mathbb{P}^{[1]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{LL}} - \frac{2\pi b_1}{b_0} \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{LL}} \right)$$

with the $\mathcal{J}_0^{\mathrm{LL}}$ and $\mathcal{J}_0^{\mathrm{NLL}}$ terms related to the integral of the initial conditions (so to avoid distributions). The coefficients of the recursive solution are found as:

$$J_k^{\mathrm{LL}}(z) = -\frac{d}{dz}\mathcal{J}_k^{\mathrm{LL}}(z)\,, \qquad J_k^{\mathrm{NLL}}(z) = -\frac{d}{dz}\mathcal{J}_k^{\mathrm{NLL}}(z)\,.$$

We calculated J_k^{LL} up to $k = 3$ and J_k^{NLL} up to $k = 2$ (i.e. up to order α^3) for non-singlet, singlet and photon PDF.

Non-singlet recursive solution up to $\mathcal{O}(\alpha^3)$

$$\hat{J}^{\text{LL}}_{\text{NS},0} = 1 - z$$

$$\hat{J}^{\text{LL}}_{\text{NS},1} = \frac{1}{z-1} \times \left(3z^2\log(z) + (1-z)(z+3) - 4(z-1)^2\log(1-z) + \log(z)\right)$$

$$\hat{J}^{\text{LL}}_{\text{NS},2} = \frac{1}{4(z-1)} \times \Big(-24\left(z^2-1\right)\text{Li}_2(z) + 24\left(-z^2+2\left(z^2+1\right)\log(z)+3\right)\log(1-z) - 2\log(z)\Big(-9z^2+7z^2\log(z)+$$
$$\log(z)-3\Big) + (z-1)\left(4\pi^2(3z-1) - 3(5z+19)\right) - 48(z-1)^2\log^2(1-z) - 48z(\log(2-2z) - \log(2z))\Big)$$

$$\hat{J}^{\text{NLL}}_{\text{NS},0} = (1-z)(L_0 - 2\log(1-z) - 1)$$

$$\hat{J}^{\text{NLL}}_{\text{NS},1} = \frac{1}{18b_0(z-1)(z+1)} \times \Big(36\pi b_0^2L_0z^3 - 36\pi b_0^2L_0z^2 - 36\pi b_0^2L_0z + 36\pi b_0^2L_0 - 36\pi b_0^2z^3 - 72\pi b_0^2z^3\log(1-$$
$$z) + 36\pi b_0^2z^2 + 72\pi b_0^2z^2\log(1-z) + 36\pi b_0^2z + 72\pi b_0^2z\log(1-z) - 72\pi b_0^2\log(1-z) - 36\pi b_0^2 - 18b_0L_0z^3 -$$
$$72b_0L_0z^3\log(1-z) + 54b_0L_0z^3\log(z) - 54b_0L_0z^2 + 72b_0L_0z^2\log(1-z) + 54b_0L_0z^2\log(z) + 18b_0L_0z +$$
$$72b_0L_0z\log(1-z) + 18b_0L_0z\log(z) - 72b_0L_0\log(1-z) + 18b_0L_0\log(z) + 54b_0L_0 + 44b_0N_Fz^3 +$$
$$12b_0N_Fz^3\log(z) - 68b_0N_Fz^2 + 12b_0N_Fz^2\log(z) - 44b_0N_Fz + 12b_0N_Fz\log(z) + 12b_0N_F\log(z) +$$
$$68b_0N_F + 72b_0\left(z^3 - z^2 - 2z + 2\right)\text{Li}_2(-z) + 216b_0z\text{Li}_2\left(\frac{1}{z+1}\right) - 36b_0(z-1)(z+1)^2\text{Li}_2(1-z) -$$
$$216b_0\text{Li}_2\left(\frac{1}{z+1}\right) + 153b_0z^3 - 6\pi^2b_0z^3 + 108b_0z^3\log^2(1-z) - 27b_0z^3\log^2(z) + 54b_0z^3\log(1-z) -$$
$$90b_0z^3\log(z) - 72b_0z^3\log(1-z)\log(z) + 72b_0z^3\log(z)\log(z+1) - 135b_0z^2 + 6\pi^2b_0z^2 - 108b_0z^2\log^2(1-$$
$$z) + 9b_0z^2\log^2(z) + 18b_0z^2\log(1-z) - 90b_0z^2\log(z) - 72b_0z^2\log(1-z)\log(z) - 72b_0z^2\log(z)\log(z+$$
$$1) - 153b_0z - 18\pi^2b_0z - 108b_0z\log^2(1-z) - 9b_0z\log^2(z) + 108b_0z\log^2(z+1) + 108b_0\log^2(1-z) +$$
$$27b_0\log^2(z) - 108b_0\log^2(z+1) - 54b_0z\log(1-z) + 72b_0z\log(z) - 144b_0z\log(z)\log(z+1) - 18b_0\log(1-$$
$$z) + 72b_0\log(z) + 144b_0\log(z)\log(z+1) + 135b_0 + 18\pi^2b_0 + 36\pi b_1z^3 - 36\pi b_1z^2 - 36\pi b_1z + 36\pi b_1\Big)$$

$$\hat{J}^{\text{NLL}}_{\text{NS},2} = \frac{1}{36b_0z\left(z^2-1\right)} \times \Big((1-z)z\Big(144(L_0-1)\pi^2\left(z^2-1\right)b_0^3 - 8\pi\Big(18L_0(z+1)(z+3) - 2N_F(z+1)(11z-$$
$$17) + 3\left(3\left(-8+\pi^2\right)z^2 - 15z + \pi^2 + 9\right)\Big)b_0^2 + \Big(2\Big(-2N_F\left(4\pi^2(z-1) - 19z + 63\right)(z+1) + 9(57z-73)(z+$$
$$1) + 6\pi^2\left(z(17z+3) + 12b_1\left(z^2-1\right) - 2\right)\Big) - 9L_0(z+1)\left(8\pi^2(z-1) - 3(5z+19)\right)\Big)b_0 - 144b_1\pi(z+1)(z+$$
$$3)\Big) + 2\Big(6b_0\log^3(z)z^4 - 18b_0\log^2(z)z^4 - 63b_0L_0\log^2(z)z^4 - 36b_0N_F\log^2(z)z^4 + 54b_0^2\pi\log^2(z)z^4 -$$

$$216b_0^2\pi(\log(1-z) - \log(z))z^4 - 342b_0\log(z)z^4 + 81b_0L_0\log(z)z^4 - 156b_0N_F\log(z)z^4 +$$
$$72b_0\log^2(2)\log(z)z^4 - 72b_0\log(2)\log(z)z^4 - 18b_0\pi^2\log(z)z^4 - 216b_1\pi\log(z)z^4 - 216b_0^2L_0\pi\log(z)z^4 -$$
$$24b_0^2N_F\pi\log(z)z^4 + 18b_0\log^3(z)z^3 + 144b_0\log^2(z)z^3 - 63b_0L_0\log^2(z)z^3 - 36b_0N_F\log^2(z)z^3 -$$
$$18b_0^2\pi\log^2(z)z^3 + 234b_0\log(z)z^3 + 297b_0L_0\log(z)z^3 - 12b_0N_F\log(z)z^3 - 72b_0\log^2(2)\log(z)z^3 +$$
$$144b_0\log(2)\log(z)z^3 + 42b_0\pi^2\log(z)z^3 + 288b_0^2\pi\log(z)z^3 - 216b_1\pi\log(z)z^3 - 216b_0^2L_0\pi\log(z)z^3 -$$
$$24b_0^2N_F\pi\log(z)z^3 - 6b_0\log^3(z)z^2 + 54b_0\log^2(z)z^2 - 9b_0L_0\log^2(z)z^2 - 12b_0N_F\log^2(z)z^2 +$$
$$18b_0^2\pi\log^2(z)z^2 + 306b_0\log(z)z^2 + 243b_0L_0\log(z)z^2 + 68b_0N_F\log(z)z^2 - 72b_0\log^2(2)\log(z)z^2 +$$
$$288b_0\log(2)\log(z)z^2 + 66b_0\pi^2\log(z)z^2 - 36b_0^2\pi\log(z)z^2 - 72b_1\pi\log(z)z^2 - 72b_0^2L_0\pi\log(z)z^2 -$$
$$24b_0^2N_F\pi\log(z)z^2 + 288b_0(z-1)^2(z+1)\log^3(1-z)z - 18b_0\log^3(z)z - 48b_0(z-4)(z-1)(z+4)\log^3(z+1)z -$$
$$144b_0\log^2(z)z - 9b_0L_0\log^2(z)z - 12b_0N_F\log^2(z)z - 54b_0^2\pi\log^2(z)z - 270b_0\log(z)z + 27b_0L_0\log(z)z -$$
$$76b_0N_F\log(z)z + 72b_0\log^2(2)\log(z)z - 144b_0\log(2)\log(z)z - 42b_0\pi^2\log(z)z - 108b_0^2\pi\log(z)z -$$
$$72b_1\pi\log(z)z - 72b_0^2L_0\pi\log(z)z - 24b_0^2N_F\pi\log(z)z - 36b_0(z+1)\log^2(1-z)\Big((6L_0(z-1) + 12b_0\pi(z-1) -$$
$$5z-7)(z-1) + \left(11z^2+1\right)\log(z)\Big)z + \log(1-z)\Big(144\pi^2(z-1)^2(z+1)b_0^3 + 72\pi\Big(4L_0(z+1)(z-1)^2 - 3z(z-1) +$$
$$3\Big)b_0^2 - (z-1)\Big(108L_0(z+1)(z+3) + 32N_F((3-8z)z+11) + 3\Big(-27z(7z+4) + 4\pi^2\left(9z^2+z-6\right) + 120\log^2(2) +$$
$$81\Big)\Big)b_0 + 12\Big(3(z-1)(z(2z+5)+1)\log^2(z) + (z+1)\Big((30\pi b_0 + 27L_0 + 4N_F - 30)z^2 - 24z + 9L_0 + 4N_F +$$
$$6b_0\pi + 27\Big)\log(z) + 24(z-1)(2\log(2) - \log(z))\log(z+1)\Big)b_0 + 288b_1\pi(z-1)^2(z+1)\Big)z - 108b_0\log^2(z) -$$
$$18b_0\Big(-16\log(2)z^4 + 8(\log(z) + 2\log(2))z^3 + 4(15z-13)\log(z)z + 8(z-1)(\log(1-z) + \log(z) + 2\log(2))z +$$
$$(z-1)(z(12\pi b_0 + z(14z+29) - 23) - 12)\Big)\log^2(z+1) + 12b_0\log(2)\Big(-4\pi^2z\left(z^3+2z-1\right) + (z-1)(z+$$
$$1)(z(9z+7)+3)\log(8) + 2z\left(z\left(4z^2+z+16\right) - 11\right)\log^2(2)\Big) - 216b_0\log(2)\log(z) - 12b_0\Big(6z(z+1)(z(2z-$$
$$3)+2)\log^2(z) + 6(z-1)z\left(-4z + 2b_0\pi\left(z^2-2\right) - 3\right)\log(z) + (z-1)\left(12z\log^2(2)\left(z^2+4\right) + \pi^2\left(z^3+z\right) +$$
$$6\Big(z\left((-1+2\log(2))z^2 + z + 5\right) - 2z(\log(1-z) + \log(2)) + 3\Big)(\log(z) + \log(2))\right)\Big)\log(z+1)\Big) + 18b_0\Big(-$$
$$16z(z+1)\text{Li}_3\left(\frac{1-z}{2}\right)(z-1)^2 + 16z(z+1)\text{Li}_3\left(\frac{z-1}{2z}\right)(z-1)^2 - 56z(z+1)\text{Li}_3\left(\frac{z}{z+1}\right)(z-1)^2 + 16z(z+$$

$$1)\text{Li}_3\left(\frac{2z}{z+1}\right)(z-1)^2 - 16z\Big((\log(z+1) - \log(1-z))z^2 + \log(1-z) - \log(z+1) - 5\log(2)\Big)\text{Li}_2\left(\frac{1-z}{2}\right)(z-$$
$$1) + 8z(z+1)(7z+4)\text{Li}_3(1-z)(z-1) + 4z(z+1)(7z+5)\text{Li}_3(z)(z-1) - 16z\left(z^2+4\right)\text{Li}_3\left(\frac{z+1}{2}\right)(z-1) +$$
$$4z(z+1)\Big((4\pi b_0 + 3L_0 - 8)z^2 + 2(z(-8\log(1-z) + \log(z) + 2\log(z+1)) - 4(\log(z+1) - \log(1-z)))z - 3L_0 +$$
$$\log\left((z-1)^8\right) - 2\log(z) + 4\log(z+1) - 4b_0\pi + 11\Big)\text{Li}_2(1-z) + 8\left(z^2-1\right)(z(2\log(2)(z-1) - z + 2) - 2(z-$$
$$1)z\log(1-z) + 3)\text{Li}_2\left(\frac{z-1}{2z}\right) - 4\Big((z-1)\Big(z\Big(z(15z-1) + 4b_0\pi\left(z^2-2\right) - 20\Big) - 6\Big) + 4z\Big(\log(z)z^3 + \log(1-$$
$$z)z - \log(1-z) + \log(z) + (z(z+6)-5)\log(z+1)\Big)\Big)\text{Li}_2(-z) + 8z\Big((z-1)\Big(4z^2 - 2z + 6\log(1-z) - 6b_0\pi -$$
$$3\Big) + 2\left(z^3+5z-4\right)\log(z+1)\Big)\text{Li}_2\left(\frac{1}{z+1}\right) - 4\left(z^2-1\right)(z(5z-8) + 10(z-1)z\log(z+1) - 6)\text{Li}_2\left(\frac{z}{z+1}\right) +$$
$$8\left(z^2-1\right)(z(2\log(2)(z-1) - z + 2) - 2(z-1)z\log(z+1) + 3)\text{Li}_2\left(\frac{2}{z+1} - 1\right) - 8z^2\left(z^2-1\right)\text{Li}_3\left(\frac{z-1}{z}\right) -$$
$$8z(z+1)(z(7z-16)+7)\text{Li}_3(-z) + 16z\left(-3z^3+z^2+z-3\right)\text{Li}_3\left(\frac{1}{z+1}\right) + z(z(z(61z-69)+59)+29)\zeta(3)\Big)\Big)$$

Analytic asymptotic solutions

Why?

- Recursive solutions are valid in the whole z range, but not good near $z = 1$ (they feature $\log^p(1-z)/(1-z)$ and $\log^p(1-z)$ terms)
- rapid growth of the electron PDF at $z \to 1$ (initial condition at LO is $\delta(1-z)$) $\to$ analytical knowledge crucial in the context of numerical computations of cross sections

How?

- Go to Mellin space:

$$M[f] \equiv f_N = \int_0^1 dz\, z^{N-1} f(z)\,.$$

- Solve the evolution equation there.
- Come back to z-space by an inverse Mellin transform.

Asymptotic behavior in Mellin space

- The evolution equation becomes multiplicative (between scalars or matrices):

$$\Gamma_N(\mu^2) = E_N(\mu^2, \mu_0^2)\,\Gamma_{0,N}(\mu_0^2)$$

- The large-z region corresponds to the large-N region:

$$P_N^{[0]} \xrightarrow{N\to\infty} -2\,\log\left(N\,e^{\gamma_E}\right) + 2\lambda_0\,, \quad P_N^{[1]} \xrightarrow{N\to\infty} \frac{20}{9} N_F\,\log\left(N\,e^{\gamma_E}\right) + \lambda_1\,.$$

LL solution for non-singlet [Gribov, Lipatov; (1972)]

$$\Gamma^{\mathrm{LL}}(z,\mu^2) = \frac{e^{-\gamma_E \eta_0}\,e^{\lambda_0 \eta_0}}{\Gamma(1+\eta_0)}\,\eta_0(1-z)^{-1+\eta_0}, \quad \eta_0 = \frac{\alpha}{\pi}\log\frac{\mu^2}{\mu_0^2}$$

We are resumming the $\log(1-z)/(1-z)$ divergent terms to all order in α.

Non-singlet asymptotic solution at NLL

$$\Gamma^{\text{NLL}}(z,\mu^2) = \frac{e^{-\gamma_E \xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1} \Bigg\{ 1 + \frac{\alpha(\mu_0)}{\pi} \Bigg[\left(\log\frac{\mu_0^2}{m^2} - 1\right)\left(A(\xi_1) + \frac{3}{4}\right) - 2B(\xi_1) + \frac{7}{4} + \left(\log\frac{\mu_0^2}{m^2} - 1 - 2A(\xi_1)\right)\log(1-z) - \log^2(1-z) \Bigg] \Bigg\}$$

with:

$$\xi_1 = 2t + \mathcal{O}(\alpha^2), \quad \hat{\xi}_1 = \frac{3}{2}t + \mathcal{O}(\alpha^2), \quad A(\xi_1) = \frac{1}{\xi_1} + \mathcal{O}(\xi_1)\,, \quad B(\xi_1) = -\frac{\pi^2}{6} + 2\zeta_3\xi_1 + \mathcal{O}(\xi_1^2)$$

- NLL still very peaked towards $z = 1$, with behavior worse than LL
- if $\mu_0 \simeq m_e$ and $\mu \simeq 100$ GeV, then $\xi_1 \simeq 0.05 \to$ the $\log(1-z)$ term is much larger than the $\log^2(1-z)$ one, even for z values *extremely* close to one.

Singlet and photon cases

Dominant term of the splitting matrices in the large-N region are:

$$\mathbb{P}_{\mathrm{S},N} \overset{N\to\infty}{\longrightarrow} \begin{pmatrix} -2\log\bar{N} + 2\lambda_0 & 0 \\ 0 & -\frac{2}{3}N_F \end{pmatrix} + \frac{\alpha}{2\pi}\begin{pmatrix} \frac{20}{9}N_F\log\bar{N} + \lambda_1 & 0 \\ 0 & -N_F \end{pmatrix} + \mathcal{O}(\alpha^2)$$

This is a diagonal matrix $\to$ independent evolution

- Singlet solution = non-singlet solution (i.e. the mixing with the photon does not affect the electron large-z behaviour)
- Photon solution:

$$\Gamma_\gamma(z,\mu^2) = \frac{\alpha(\mu_0)}{\alpha(\mu)}\left[\frac{\alpha(\mu_0)}{2\pi}\frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right)\right]$$

Unfortunately, this equation does not work . . . here mixing is important!

Improvement of photon large-z PDF

Solving the evolution equations by including off-diagonal elements implies a significant increase in complexity. Main idea: solve the matrix differential equation by treating the off-diagonal subdominant terms ($\mathcal{O}(1/N)$) as a "perturbation" of the diagonal result ($\log \bar{N}$ and constants):

$$\mathbb{E}_N(t) = \mathbb{E}_N^{(0)}(t)\,\mathbb{E}_N^{(1)}(t).$$

Then convolve with initial conditions and perform the Mellin anti-trasform.
The final result is rather involved, but dominant terms in the $z \to 1$ limit are:

$$\Gamma_\gamma(z) \xrightarrow{z\to 1} \frac{\alpha(\mu_0)}{\alpha(\mu)}\left[\left(\frac{\alpha(\mu_0)}{2\pi}\right)\frac{3}{\xi_{1,0}}\log(1-z) - \left(\frac{\alpha(\mu_0)}{2\pi}\right)^2\frac{1}{2\,\xi_{1,0}}\log^3(1-z)\right]$$

where $\xi_{1,0} = 2 + \mathcal{O}(\alpha)$. Formally dominant term suppressed w.r.t the subdominant one by a factor proportional to α.

Consistency checks and matching

Once we have both the recursive and the asymptotic solutions we can check if:

- ✓ the expansion of the asymptotic solutions reproduce order by order the most singular terms of the recursive solutions;
- ✓ the asymptotic solutions reproduce the large-z limit of the initial conditions for $\mu \to \mu_0$ (verified at the distributional level).

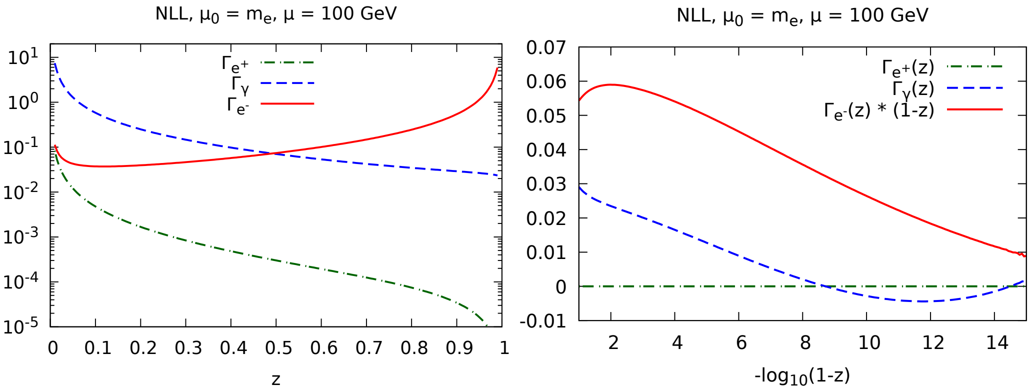
Finally, we combine the recursive and asymptotic solutions by means of an additive formula:

$$\Gamma_{\text{mtc}}(z) = \Gamma_{\text{rec}}(z) + \Big(\Gamma_{\text{asy}}(z) - \Gamma_{\text{subt}}(z)\Big) G(z)\,,$$

with Γ_{subt} chosen as $\mathcal{O}(\alpha^3)$ expansion of Γ_{asy}, and $G \equiv 1$ for NS/S (non trivial G required in the case of the photon, see backup).

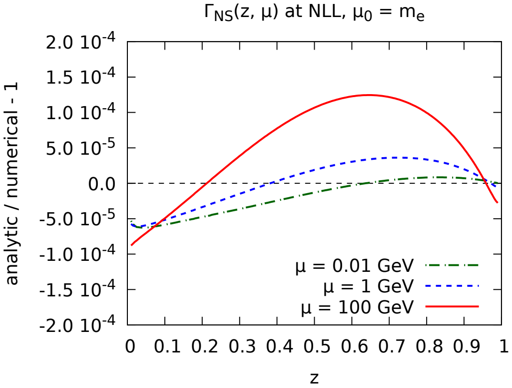
Outline

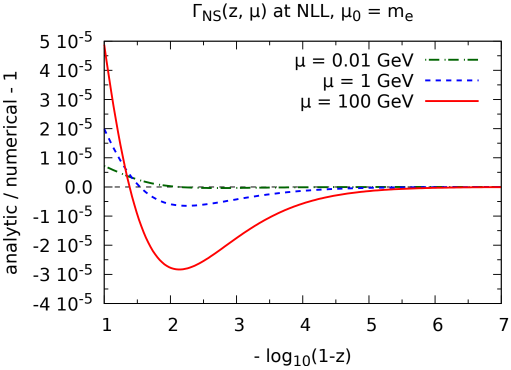
Electron PDFs at NLL



Electron still dominates at large z, while photon at small z (however, remember the constraint $z_+ z_- \geq M^2/s$ for the production of object with mass M^2 in the final state).

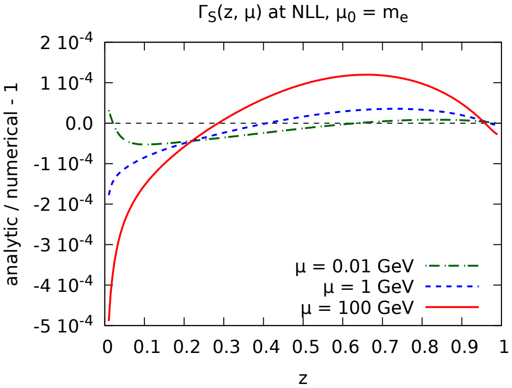
Analytical vs. numerical (non-singlet)

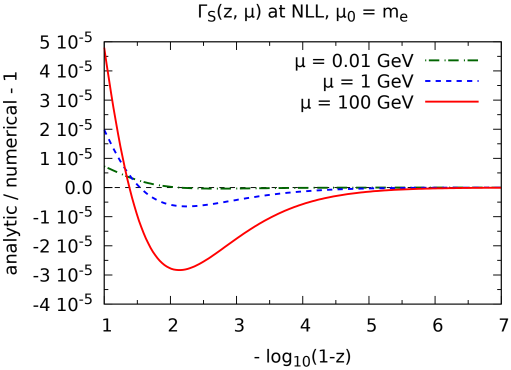




Worst-case scenario ($\mu = 100$ GeV): agreement at the $10^{-4} - 10^{-5}$ level

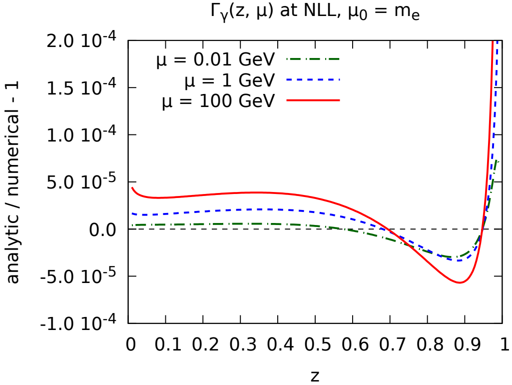
Analytical vs. numerical (singlet)

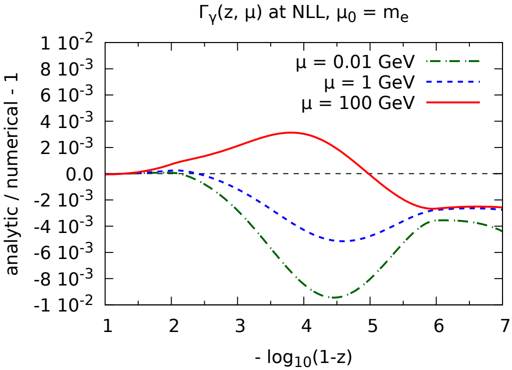




On the linear scale, largest discrepancy at small z's for the singlet

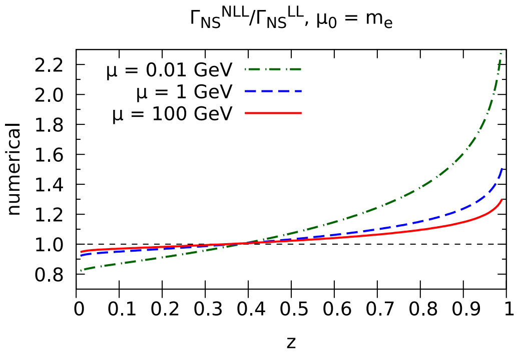
Analytical vs. numerical (photon)

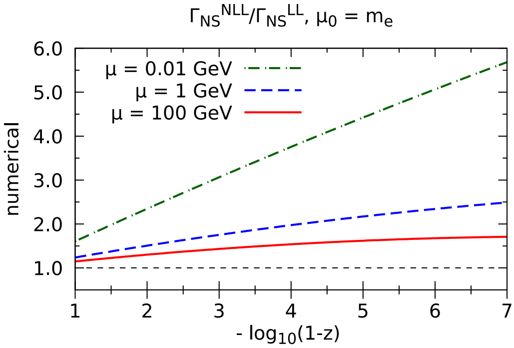




Photon problematic on the log scale, but small in absolute value

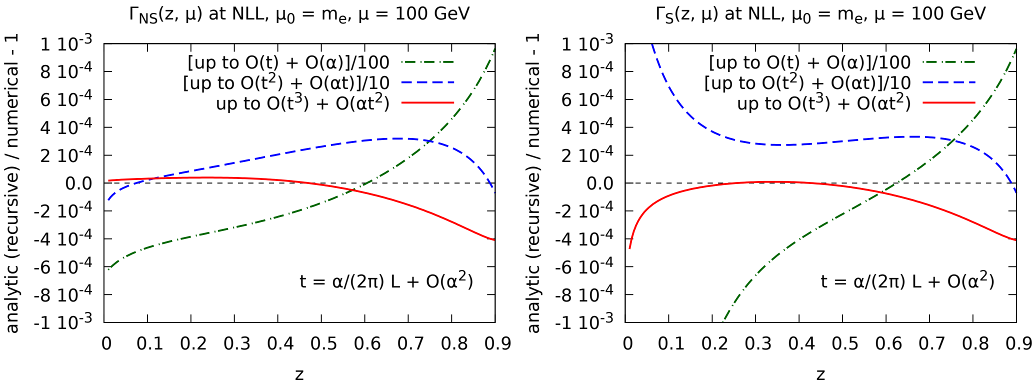
NLL vs. LL (non-singlet)



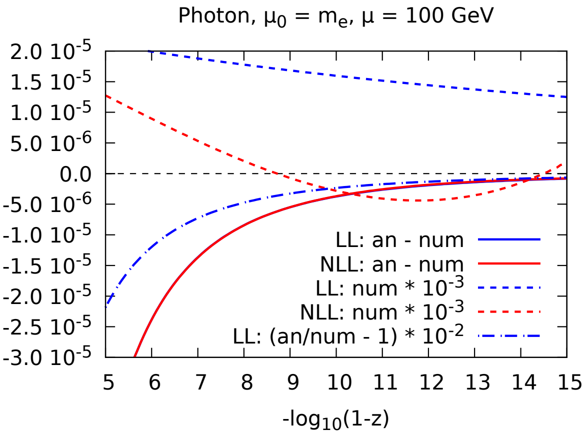


$$\Gamma^{\rm NLL}_{\rm NS/S}(z,\mu^2) \sim {\rm LL}\left(1+\frac{\alpha(\mu_0)}{\pi}\left[a+\frac{b}{\alpha(\mu)\log(\mu^2/\mu_0^2)}\log(1-z)-\log^2(1-z)\right]\right)$$

Convergence of analytical recursive solutions



Asymptotic photon behaviour



Onset of the true asymptotic regime occurs at much larger z values!

Outline

QED Parton Distribution Functions

Analytical solutions

Results in the $\overline{\text{MS}}$ scheme

Change of subtraction scheme

Summary

PDFs and short-distance cross section in $\overline{\text{MS}}$

$$K_{ee}(z) = K_{\gamma e}(z) = 0 \quad \Longleftrightarrow \quad \overline{\text{MS}}\,.$$

- The electron PDF initial condition features an explicit $\log(1-z)$ term, whose presence is a by-product of the subtraction procedure, but it is not physically motivated.
- The asymptotic large-z solutions feature explicit $\log^p(1-z)$ terms. It's yet unclear whether the origin of these terms is in part or entirely an artifact of the $\overline{\text{MS}}$ scheme.
- The scheme-changing terms K_{ee} and $K_{\gamma e}$ enter also the short-distance cross section (in FKS formalism through the degenerate $(n+1)$-body contribution):

$$d\hat{\bar{\sigma}}^{(3)}_{e^+e^-}(p_{e^+},p_{e^-}) \propto \frac{\alpha}{2\pi}\Bigg\{ e_e^2\left(1+(1-\xi)^2\right)\left[\left(\frac{1}{\xi}\right)_+ \log\frac{s}{\mu^2} + 2\left(\frac{\log\xi}{\xi}\right)_+\right] + e_e^2\xi^2\left(\frac{1}{\xi}\right)_+ - e_e^2 \underbrace{K_{ee}(1-\xi)}_{\equiv 0 \text{ in } \overline{\text{MS}}}\Bigg\} d\xi\,.$$

Delta scheme

The choice of a particular scheme should **not** affect the value of physical cross sections (after convolution) i.e. the difference between results in difference schemes should be of higher order.

However, motivated by the issue pointed out in the $\overline{\text{MS}}$ case, we are also exploring a scheme where the initial conditions are maximally simplified (similar to DIS scheme in QCD):

$$\Gamma_{e^-}^{[1],(\Delta)}(z,\mu_0^2) = \log\frac{\mu_0^2}{m^2}\left[\frac{1+z^2}{1-z}\right]_+$$

$$\Gamma_{\gamma}^{[1],(\Delta)}(z,\mu_0^2) = \log\frac{\mu_0^2}{m^2}\left(\frac{1+(1-z)^2}{z}\right)$$

$$\Gamma_{e^+}^{[1],(\Delta)}(z,\mu_0^2) = 0$$

Outline

Take-home messages

- We computed the electron, positron and photon PDFs of the unpolarised e^- at NLL (and by charge conjugation also the ones of the incoming e^+).
- They are obtained by means of both **numerical and analytical methods, which agree extremely well** in the region relevant for phenomenology.
- In the $\overline{\text{MS}}$ scheme, **the large-z peak is even more pronunced than at LL** (this is in part an artefact of that scheme).
- Hence, **analytical knowledge of NLL large-z electron PDF is crucial** when performing convolutions.
- NLL electron PDFs not only provide a NLL correction to processes with incoming electrons, but also allow to treat **photon-initiated hard processes in the same framework**.

Next steps

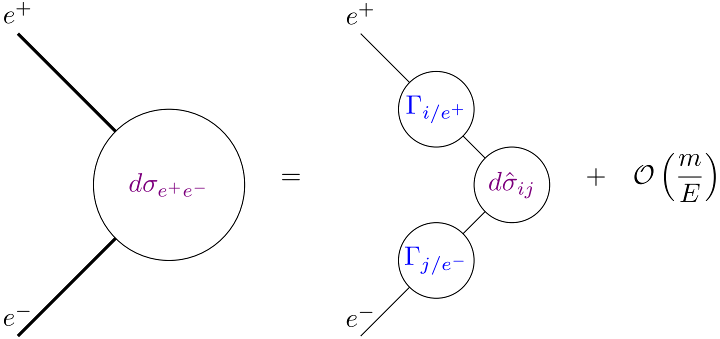
Final goal:
Establish the phenomenological impact of NLL PDFs (in both schemes) on observables:

- In the Delta scheme, in order to have a solution valid in the whole z-range, we have designed a smooth switch between the numerical solution and the large-z analytical solution.
- Ongoing tests with a toy model cross section ($e^+e^- \to u\bar{u}(\gamma)$, with only initial state radiation). Numerical convolution with short-distance cross section turns out to be highly non trivial, in particular when dealing with plus prescriptions.
- We plan in a near future to study more realistic cross sections, by interfacing our PDFs with a NLO event generator.

Thanks for your attention!

Backup slides

QED factorisation formula



Evolution operator formalism

In Mellin space, we can introduce the evolution operator:

$$\Gamma_N(\mu^2) = \mathbb{E}_N(\mu^2, \mu_0^2)\,\Gamma_{0,N}\,, \quad \mathbb{E}_N(\mu_0^2, \mu_0^2) = \mathbb{I}$$

Convenient to introduce a variable t:

$$t = \frac{1}{2\pi b_0} \log \frac{\alpha(\mu)}{\alpha(\mu_0)} = \frac{\alpha(\mu)}{2\pi} \log \frac{\mu^2}{\mu_0^2} + \mathcal{O}(\alpha^2)$$

i.e. LL terms: t^k and NLL terms: $(\alpha/(2\pi))\,t^k$. We obtain:

$$\frac{\partial \mathbb{E}_N(t)}{\partial t} = \left[\mathbb{P}_N^{[0]} + \frac{\alpha(\mu)}{2\pi}\left(\mathbb{P}_N^{[1]} - \frac{2\pi b_1}{b_0}\mathbb{P}_N^{[0]}\right)\right]\mathbb{E}_N(t) + \mathcal{O}(\alpha^2)$$

which can be solve analytically (NS) or numerically (S/γ).

Matching

Combine the recursive and the asymptotic solution by means of an additive formula:

$$\Gamma_{\rm mtc}(z) = \Gamma_{\rm rec}(z) + \Big(\Gamma_{\rm asy}(z) - \Gamma_{\rm subt}(z)\Big) G(z)\,, \quad \lim_{z\to 1} G(z) = 1$$

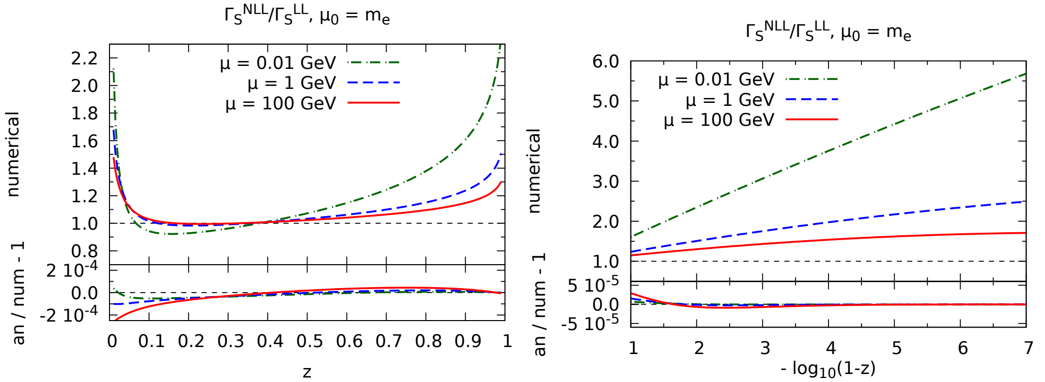
Choice of subtraction term $\Gamma_{\rm subt}$ and matching function G dictated by:

$$\Gamma_{\rm mtc} \sim \Gamma_{\rm asy} \quad z \simeq 1$$
$$\Gamma_{\rm mtc} \sim \Gamma_{\rm rec} \quad \text{small- and intermediate-}z$$

After technical studies:

- $\Gamma_{\rm subt}$ chosen as $\mathcal{O}(\alpha^3)$ expansion of $\Gamma_{\rm asy}$
- different strategy for G:
  - NS/S: $G(z) \equiv 1$ ($\Gamma_{\rm asy}(z) - \Gamma_{\rm subt}(z)$ cancel very well in the small-z region)
  - γ: non trivial G needed ($\Gamma_{\rm asy}$ problematic in the small-z region)! $G(\hat{z}_0, \hat{z}_1, p)$ (transition between $\Gamma_{\rm rec}$ and $\Gamma_{\rm asy}$ in the region $\hat{z}_0 < -\log_{10}(1-z) < \hat{z}_1$ with p used to adjust the abruptness of the transition)

NLL vs. LL (singlet)



NLL vs. LL (photon)