Lecture 3: Non-linear-sigma-model: Part I

Christopher Mudry*

Paul Scherrer Institut, CH-5232 Villigen PSI, Switzerland.

(Dated: March 07, 2011)

Abstract

The $O(N)$ non-linear-sigma-model (NLSM) is defined as an effective field theory that encodes the pattern of symmetry breaking of an $O(N)$ classical Heisenberg model defined on a square lattice. A geometric and group theoretical interpretation of the $O(N)$ NLSM is given. The notions of fixed points, relevant, marginal, and irrelevant perturbations are introduced. The real-valued scalar (free) field theory in 2d is taken as an example of a critical field theory and the Callan-Symanzik equations for $(m + n)$ point correlation functions are derived. The Callan-Symanzik equations are generalized to include relevant coupling constants. The spin-spin correlator is evaluated in an expansion in the $O(N > 2)$ NLSM coupling constant in 2d.
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I. INTRODUCTION

The so-called non-linear-sigma-models (NLSM) will occupy us for the next two lectures. NLSM were first introduced in high-energy physics in the context of chiral symmetry breaking. NLSM also play an essential role in condensed matter physics where they appear naturally as effective field theories describing the low-energy long-wave-length limit of numerous microscopic models. I begin with some examples of NLSM.

II. NON-LINEAR-SIGMA-MODELS (NLSM)

A. Definition of $O(N)$–NLSM

An example for an unfrustrated classical $O(N)$ Heisenberg magnet is given by the classical partition function

$$Z_{N,\beta,H} := \int \mathcal{D}[S] \exp \left( (-)^2 \beta \left( \sum_{\langle ij \rangle} J S_i \cdot S_j + \sum_{i=1}^{N} H \cdot S_i \right) \right),$$

(2.1a)

$$S_i = S_{i+L \hat{e}_{\mu}} \in \mathbb{R}^N, \quad S_i^2 = 1, \quad i \in \mathbb{Z}^d, \quad H \in \mathbb{R}^N, \quad N \in \mathbb{N},$$

(2.1b)

$$\hat{e}_\mu = (0 \cdots 0 1 0 \cdots 0), \quad \mu = 1, \cdots, d.$$  

(2.1c)

On each site $i \in \mathbb{Z}^d$ of a $d$-dimensional hypercubic lattice made of $N = L^d$ sites, a unit vector $S_i$ of $\mathbb{R}^N$ interacts with its $2d$ nearest-neighbors $S_{i \pm \hat{e}_\mu}, \mu = 1, \cdots, N, \quad \langle ij \rangle \equiv \langle i(i + \hat{e}_\mu) \rangle$ through the ferromagnetic Heisenberg coupling constant (also called spin stiffness) $J > 0$ as well as with an external uniform magnetic field $H$. Periodic boundary conditions are imposed. The measure $\mathcal{D}[S]$ is the product measure for the infinitesimal surface element of the unit sphere in $\mathbb{R}^N$. Alternatively, one could write

$$Z_{N,\beta,H} = \left[ \prod_{i=1}^{N} \int_{-\infty}^{+\infty} dN S_i \right] \delta \left( \frac{1}{g^2} - S_i^2 \right) \exp \left( a^{-2} \sum_{\langle ij \rangle} S_i \cdot S_j + a^{d} \sum_{i=1}^{N} h \cdot S_i \right),$$

(2.2a)

or

$$Z_{N,\beta,H} = \lim_{\lambda \to \infty} \left( \prod_{i=1}^{N} \int_{-\infty}^{+\infty} dN S_i \right) \exp \left( a^{-2} \sum_{\langle ij \rangle} S_i \cdot S_j + a^{d} \sum_{i=1}^{N} h \cdot S_i - \lambda \left( \frac{1}{g^2} - S_i^2 \right)^2 \right),$$

(2.2b)
or

\[
Z_{N,\beta,H} = \left( \prod_{i=1}^{N} \int_{-\infty}^{+\infty} dN_i \int_{-\infty}^{+\infty} d\lambda_i \right) \exp \left( a^{d-2} \sum_{\langle ij \rangle} s_i \cdot s_j + a^d \sum_{i=1}^{N} \left[ h \cdot s_i + i\lambda_i \left( \frac{1}{g^2} - s_i^2 \right) \right] \right),
\]

(2.2c)

where

\[
g^2 := a^{d-2} \frac{1}{\beta J}, \quad h := a^{-d} \frac{2}{d+2} \sqrt{\frac{\beta}{J}} \mathbf{H} = a^{-d} \beta g \mathbf{H}.
\]

(2.2d)

Here, the lattice spacing \( a \) is introduced in anticipation of taking the continuum limit.

Both the measure and the Heisenberg interaction are invariant under any global rotation \( Q \in O(N) \) of the spins \( \mathbf{S}_i \)

\[
\mathbf{S}_i = R \tilde{\mathbf{S}}_i, \quad \forall i.
\]

(2.4)

The uniform magnetic field \( \mathbf{H} \) breaks this global \( O(N) \) invariance down to the subgroup \( O(N-1) \) of global rotations in the \( (N-1) \)-dimensional subspace of \( \mathbb{R}^N \) orthogonal to \( \mathbf{H} \) (see Fig. 1). A magnetic field should here be thought of as either a formal device to break the \( O(N) \) symmetry down to \( O(N-1) \) when it is uniform or as a source term inserted for mathematical convenience to compute correlation functions in which case it can be taken to be nonuniform. In both interpretations, it must be set to zero at the end of the day [see Eq. (2.12)].

Invariance of the partition function under the transformation

\[
J \rightarrow -J, \quad \mathbf{S}_i \rightarrow (-)^{|i|} \mathbf{S}_i, \quad \mathbf{H} \rightarrow (-)^{|i|} \mathbf{H}, \quad |i| \equiv \sum_{\mu=1}^{d} i_{\mu}, \quad i = \sum_{\mu=1}^{d} i_{\mu} \epsilon_{\mu}
\]

(2.5)

defines absence of frustration. In general, a lattice is said to be geometrically frustrated when it cannot be decomposed into two interpenetrating sublattices. For frustrated lattices,

\footnote{The group of orthogonal matrices \( O(N) \) is made of all \( N \times N \) matrices \( R \) with real-valued matrix elements, nonvanishing determinant, and obeying

\[
R^t R = R R^t = \mathbb{1}_N.
\]

Equation (2.3) implies that the determinant of an orthogonal matrix is \( \pm 1 \). The subgroup \( SO(N) \subset O(N) \) is made of all orthogonal matrices with determinant one.}
FIG. 1: Heisenberg model on a square lattice with $O(3)/O(2)$ symmetry in the presence of a uniform magnetic field $H$. A symmetry breaking magnetic field enforces the pattern of spontaneous-symmetry breaking into the ferromagnetic state. The symmetry group of the Heisenberg exchange interaction is $O(3)$. A uniform magnetic field $H$ breaks the $O(3)$ symmetry down to the subgroup $O(2)$ of rotations about the axis pointing along $H$. There is a one-to-one correspondence between elements of the coset space $O(3)/O(2)$ and points of the two-sphere, i.e., the surface of the unit sphere in $\mathbb{R}^3$.

Ferromagnetic ($J > 0$) and antiferromagnetic ($J < 0$) couplings are not equivalent. A next-nearest-neighbor coupling constant on a square lattice is another way by which frustration can be established.

The classical ground state of the system has all $N$ spins parallel to the external magnetic field $H$, i.e., is fully polarized into the ferromagnetic ground state. The uniform magnetization

$$M := \frac{1}{N} \sum_{i=1}^{N} S_i$$

is maximal in magnitude in the ferromagnetic ground state,

$$|M| \leq |M_{\text{ferro}}| = 1, \quad M_{\text{ferro}} := \frac{H}{|H|}$$

The expectation value of the magnetization

$$\lim_{|H| \to 0} \langle M \rangle_{Z_{N,\beta,H}} := \lim_{|H| \to 0} \frac{1}{N} \beta \partial_H \ln Z_{N,\beta,H}$$
vanishes for any finite $N$. In the thermodynamic limit, the expectation value of the magnetization (2.8) depends crucially on the order in which the two limits $N \to \infty$ and $|H| \to 0$ are taken. On the one hand, if the limit $|H| \to 0$ is taken before the thermodynamic limit $N \to \infty$, then the expectation value of the magnetization vanishes at any temperature. On the other hand, if the limit $|H| \to 0$ is taken after the thermodynamic limit $N \to \infty$, then the expectation value of the magnetization need not vanish anymore (the answer will depend on the dimensionality $d$ of the lattice), since any two configurations $\{S_i\}_{i \in \mathbb{Z}^d}$ and $\{\tilde{S}_i\}_{i \in \mathbb{Z}^d}$ of the spins that differ by a global or rigid rotation $Q \neq 1 \in O(N)$ of all spins,

$$S_i = Q \tilde{S}_i, \quad \forall i \in \mathbb{Z}^d,$$

(2.9)
differ in energy by the infinitely high potential barrier

$$\lim_{N \to \infty} [N \times |H| \cdot (1_N - Q) M|] = \infty.$$  

(2.10)

How should one decide if spontaneous symmetry breaking at zero temperature as defined by

$$1 = \lim_{|H| \to 0} \lim_{N \to \infty} \lim_{\beta \to \infty} \langle M \rangle_{Z_N,\beta, H}, \quad 0 = \lim_{N \to \infty} \lim_{|H| \to 0} \lim_{\beta \to \infty} \langle M \rangle_{Z_N,\beta, H},$$

(2.11)

extends to finite temperature, i.e.,

$$0 \neq \lim_{|H| \to 0} \lim_{N \to \infty} \langle M \rangle_{Z_N,\beta, H}, \quad 0 = \lim_{N \to \infty} \lim_{|H| \to 0} \langle M \rangle_{Z_N,\beta, H}?$$

(2.12)

Since the thermodynamic limit must matter for spontaneous-symmetry breaking to take place, we can limit ourselves to very long wave lengths. Since we want to know whether or not zero-temperature spontaneous-symmetry breaking is destroyed by thermal fluctuations for arbitrarily small temperatures, we can limit ourselves to low energies. If we are after some sort of perturbation theory, the dimensionless bare coupling constant

$$a^{-(d-2)} g^2 = \frac{1}{\beta J},$$

(2.13)

This is so because only the magnitude of the magnetization if fixed in the ferromagnetic ground state when the external magnetic field has been switched off. The direction in which the magnetization points is arbitrary. Hence, the path integral over all spin configurations can be restricted to a path integral over all spins pointing to the northern hemisphere of the $d$-dimensional unit sphere in some arbitrarily chosen spherical coordinate system provided $-M$ is added to $+M$ between the brackets on the left-hand side of Eq. (2.8).
might be a good candidate at very low temperatures and very large spin stiffness $J$. As the simplest possible effective field theory sharing the global $O(N)$ symmetry of Eqs. (2.1) or (2.2) in the absence of a symmetry breaking external magnetic field, we might try the Euclidean field theory

$$Z_{\beta, H} := \int D[n] \delta (1 - n^2) \exp \left( -\frac{1}{2g^2} \int \mathbb{R}^d d^d x \left[ (\partial_\mu n)^2 - 2a^{-d} g^2 H \cdot n \right] \right)$$

$$\propto \int D[m] \delta \left( \frac{1}{g^2} - m^2 \right) \exp \left( -\frac{1}{2} \int \mathbb{R}^d d^d x \left[ (\partial_\mu m)^2 - 2g H \cdot m \right] \right)$$

$$= \lim_{\lambda \to \infty} \left[ \prod_{x \in \mathbb{R}^d} \int_{-\infty}^{+\infty} d^N m(x) \right] e^{-\frac{1}{2} \int \mathbb{R}^d d^d x \left\{ (\partial_\mu m)^2 - 2\left[ m \cdot \lambda \left( \frac{1}{g^2} - m^2 \right) \right] \right\}}$$

(2.14)

that defines the $O(N)$ NLSM when $H = 0$.

**B. $O(N)$–NLSM as a field theory on a Riemannian manifold**

To better understand the relationship between the continuum theory (2.14) and the lattice theory (2.2), choose a coordinate system of $\mathbb{R}^N$ in which the symmetry breaking magnetic field $h$ is aligned along the direction $\hat{e}_1$ (see Fig. 1),

$$h \cdot m = |h|m_1,$$  

(2.15)
and observe that\(^3\)

\[
\int \mathcal{D}[m]\delta\left(\frac{1}{g^2} - m^2\right) \cdots =
\left\{ \prod_{x \in \mathbb{R}^d} \int_{-\infty}^{+\infty} d[m_1(x)] \cdots \int_{-\infty}^{+\infty} d[m_N(x)] \delta\left(\frac{1}{g^2} - \sum_{i=1}^{N} m_i^2(x)\right) \right\} \cdots =
\left\{ \prod_{x \in \mathbb{R}^d} \int_{-\infty}^{+\infty} d[m_1(x)] \cdots \int_{-\infty}^{+\infty} d[m_N(x)] \Theta\left(\frac{1}{g^2} - \sum_{j=2}^{N} m_j^2(x)\right) \delta\left(\frac{1}{g^2} - \sum_{j=2}^{N} m_j^2(x) - m_1^2(x)\right) \right\} \cdots +
\left\{ \prod_{x \in \mathbb{R}^d} \int_{-\infty}^{+\infty} d[m_2(x)] \cdots \int_{-\infty}^{+\infty} d[m_N(x)] \Theta\left(\frac{1}{g^2} - \sum_{j=2}^{N} m_j^2(x)\right) \frac{1}{2m_1(x)} \right\} \cdots \mid_{m_1(x) = + \sqrt{\frac{1}{g^2} - \sum_{j=2}^{N} m_j^2(x)}}^{
\left\{ \prod_{x \in \mathbb{R}^d} \int_{-\infty}^{+\infty} d[m_2(x)] \cdots \int_{-\infty}^{+\infty} d[m_N(x)] \Theta\left(\frac{1}{g^2} - \sum_{j=2}^{N} m_j^2(x)\right) \frac{1}{2|m_1(x)|} \right\} \cdots \mid_{m_1(x) = - \sqrt{\frac{1}{g^2} - \sum_{j=2}^{N} m_j^2(x)}}}
\right.
\right.
\] 

(2.17)

where \(\Theta(x)\) is the Heaviside function. We shall restrict all the local configurations \(m(x)\) entering in the path integral (2.14) to configurations called \textit{spin-waves} which are defined by the conditions that:

\(^3\) Use \(\delta(x^2 - a^2) = \delta([x - a](x + a)) = \sum_{\pm \frac{1}{a}} \delta[x - (\pm a)]\). Note also that

\[
\pi = \int_{0}^{2\pi} d\varphi \int_{0}^{\infty} dr \, r \, \delta(1 - r^2)
= \int_{\mathbb{R}^2} dx \, dy \, \delta(1 - x^2 - y^2)
= \left\{ \int_{-1}^{+1} dy \right\} \int_{-\infty}^{+\infty} dx \, \delta(1 - x^2 - y^2) + \left\{ \int_{-\infty}^{-1} dx \right\} \int_{-\infty}^{+\infty} dy \, \delta(y^2 - 1 + x^2)
\right.\left.\mid_{=0} \right.
= \left\{ \int_{-1}^{+1} \frac{dy}{2\sqrt{1 - y^2}} \right\} + \left\{ \int_{-1}^{+1} \frac{dy}{2\sqrt{1 - y^2}} \right\}
= 2\arcsin(1).
\] 

(2.16)
1. Local longitudinal fluctuations $\sigma(x)$ about the ferromagnetic state

$$m(x) = \frac{1}{g} \hat{e}_1, \quad \forall x, \quad \hat{e}_1 = (1 \ 0 \ \cdots \ 0),$$

are strictly positive

$$0 < \sigma(x) \equiv m_1(x) \leq 1/g, \quad \forall x.$$ (2.19)

2. Local transverse fluctuations $\pi_i$

$$\pi_i(x) \equiv m_{i+1}(x), \quad i = 1, \cdots, N-1,$$ (2.20)

about the ferromagnetic state are smaller in magnitude than $1/g$,

$$0 < \sigma(x) = +\sqrt{\frac{1}{g^2} - \pi^2(x)} \leq \frac{1}{g}.$$ (2.21)

3. Transverse fluctuations $\pi$ are smooth, i.e., the Taylor expansion

$$\pi_i(x + y \hat{e}_\mu) = \sum_{l=0}^{\infty} \frac{1}{l!} \left( \partial_\mu^l \pi_i \right)(x) y^l, \quad i = 1, \cdots, N-1, \quad \mu = 1, \cdots, d,$$ (2.22)

converges very rapidly when $|y|$ is of the order of the lattice spacing $a$.

Only the northern-half hemisphere of the surface of the sphere with radius $1/g$ in $\mathbb{R}^N$ is thus parametrized in the spin-wave approximation. Accessing configurations of spins in which the field $m$ points locally, say at $x$, towards the southern hemisphere, $m_1(x) < 0$, is impossible within the spin-wave parametrization (2.21). In the spin-wave approximation, the last line on the right hand side of Eq. (2.17) is neglected. This approximation is good energetically since configurations of spins in which $m_1(x)$ is negative over some finite region $\Omega$ of $\mathbb{R}^d$ is suppressed by the exponential factor of order $\exp \left[ -2|h| \int_{\Omega} d^d x \ m_1(x) \right]$ in $(\cdots)$ of Eq. (2.17). However, this argument fails to account for the entropy of the excursions of $m_1$ into the southern hemisphere, i.e., the multiplicity of spin configurations that are suppressed by an exponentially small penalty in energy for pointing anti-parallel to $h$ in some region of $\mathbb{R}^d$. The spin-wave approximation breaks down whenever the entropy of defects by which $m_1$ is anti-parallel to $h$ overcomes the energy loss.
In the spin-wave approximation, the Euclidean action of the NLSM becomes

\[
S_{\text{sw},h} := \frac{1}{2} \int_{\mathbb{R}^d} d^d x \left[ (\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2 - 2|\mathbf{h}| \sigma \right]
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^d} d^d x \left[ \frac{1}{g^2 - \pi^2} \left( \pi \partial_\mu \pi \right) \partial_\mu \pi - 2|\mathbf{h}| \sqrt{ \frac{1}{g^2 - \pi^2} } \right]
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^d} d^d x \left[ \partial_\mu \pi_i \left( \frac{\pi_i \pi_j}{g^2 - \pi^2} + \delta_{ij} \right) \partial_\mu \pi_j - 2|\mathbf{h}| \sqrt{ \frac{1}{g^2 - \pi^2} } \right]
\]

\[
\equiv \frac{1}{2} \int_{\mathbb{R}^d} d^d x \left[ (\partial_\mu \pi_i) g_{ij} (\partial_\mu \pi_j) - 2|\mathbf{h}| \sqrt{ \frac{1}{g^2 - \pi^2} } \right], \quad (2.23a)
\]

where the symmetric (metric) tensor

\[
g_{ij}(x) := \frac{g^2(\pi_i \pi_j)(x)}{1 - g^2 \pi^2(x)} + \delta_{ij}, \quad i, j = 1, \cdots, N - 1, \quad (2.23b)
\]

has been introduced and summation over repeated indices is understood. The metric tensor transforms according to

\[
(\partial_\mu \pi_i) \ g_{ij} (\partial_\mu \pi_j) = (\partial_\mu \pi_k) \tilde{g}_{kl} (\partial_\mu \pi_l), \quad (2.24a)
\]

\[
\tilde{g}_{kl} = R_{ik} g_{ij} R_{jl} = R^l_{\ kl} g_{ij} R_{jl}, \quad k, l = 1, \cdots, N - 1, \quad (2.24b)
\]

under the global rotation \( R \in O(N - 1) \) of the transverse modes \( \pi \) under which

\[
\pi(x) = R \pi(x). \quad (2.24c)
\]

In matrix form, Eq. (2.24) reads

\[
\bar{g}(x) = R^t g(x) R, \quad \forall R \in O(N - 1) \iff g(x) = R \bar{g}(x) R^t, \quad \forall R \in O(N - 1). \quad (2.25)
\]

A useful invariant under global \( O(N - 1) \) rotations of the transverse modes \( \pi \) is the determinant of the metric tensor (2.23b),

\[
\det[g(x)] = \det \left[ R \tilde{g}(x) R^t \right]
\]

\[
= \det(R) \det[\bar{g}(x)] \det(R^t)
\]

\[
= [\det(R)]^2 \det[\bar{g}(x)]
\]

\[
= \det[\bar{g}(x)], \quad \forall R \in O(N - 1). \quad (2.26)
\]
Equation (2.26) also extends to the situation when the matrix \( R \in O(N - 1) \) is allowed to vary in space, although it should then be remembered that Eq. (2.24) does not hold anymore. This observation is useful in that it allows to compute \( \det[g(x)] \) by choosing the local rotation \( R(x) \in O(N - 1) \) that rotates \( \pi(x) \) along \( \hat{e}_2 \), say, in which case \( g(x) \) is purely diagonal with the eigenvalue 1 \((N - 2)\)-fold degenerate and the eigenvalue \( \frac{\pi^2}{(N - 2)} + 1 \). Thus, we infer that

\[
\det[g(x)] = \frac{1}{1 - g^2 \pi^2(x)}, \quad \forall x \in \mathbb{R}^d. \tag{2.27}
\]

We are now ready to write in a compact manner the spin-wave approximation

\[
Z_{sw, \beta, h} := \left( \prod_{x \in \mathbb{R}^d} \int_{\mathbb{R}^{N-1}} \frac{d^{N-1}\pi(x)}{2} \sqrt{g^2 \det g(x)} \Theta \left( \left[g^2 \det g(x)\right]^{-1} \right) \right) \times \exp \left( -\frac{1}{2} \int_{\mathbb{R}^d} d^d x \left[ \left( \partial_\mu \pi_i \right) g_{ij} \left( \partial_\mu \pi_j \right) - 2|h| \left( g^2 \det g \right)^{-1/2} \right] \right) \tag{2.28a}
\]

to the partition function of the \( O(N) \)-NLSM,

\[
Z_{\beta, h} := \left( \prod_{x \in \mathbb{R}^d} \int_{\mathbb{R}^{N-1}} \frac{d^{N-1}\pi(x)}{2} \sqrt{g^2 \det g(x)} \Theta \left( \left[g^2 \det g(x)\right]^{-1} \right) \right) \times \exp \left( -\frac{1}{2} \int_{\mathbb{R}^d} d^d x \left[ \left( \partial_\mu \pi_i \right) g_{ij} \left( \partial_\mu \pi_j \right) - 2|h| \left( g^2 \det g \right)^{-1/2} \right] \right) + \left( \prod_{x \in \mathbb{R}^d} \int_{\mathbb{R}^{N-1}} \frac{d^{N-1}\pi(x)}{2} \sqrt{g^2 \det g(x)} \Theta \left( \left[g^2 \det g(x)\right]^{-1} \right) \right) \times \exp \left( -\frac{1}{2} \int_{\mathbb{R}^d} d^d x \left[ \left( \partial_\mu \pi_i \right) g_{ij} \left( \partial_\mu \pi_j \right) + 2|h| \left( g^2 \det g \right)^{-1/2} \right] \right). \tag{2.28b}
\]

The relationship between Eq. (2.14) or Eq. (2.28b) and Eq. (2.2) is that the spin-wave approximation (2.28a) on the field theory Eq. (2.14) or Eq. (2.28b) should be equivalent to the naive continuum limit of Eq. (2.2) whereby an expansion up to Gaussian order in the smooth transverse fluctuations about the ferromagnetic ground state is performed. What is left out from the naive continuum limit is the possibility that “singular” lattice configurations of the spins matter in the thermodynamic limit.\(^4\) It turns out that singular lattice

\(^4\) A configuration \( \{s_i\}_{i=1}^N \) of spins on the lattice is said to be singular if its naive continuum limit counterpart
configurations of the spins are essential to the understanding of the phase diagram of the $O(2)$–NLSM in $d = 2$ as we shall see in the lecture devoted to the Kosterlitz-Thouless transition. The usefulness of the spin-wave approximation is that it allows for an answer to the question of whether thermal fluctuations in the form of spin-waves are sufficient to destroy spontaneous-symmetry breaking at zero temperature.

Representation (2.28b) of the $O(N)$–NLSM is geometric in nature. The $(N - 1)$–sphere is an example of a Riemannian manifold on which a special choice of coordinates system, encoded by the metric (2.23b), has been made. The action in representation (2.28b) is covariant under a change of coordinate system of the $(N - 1)$-sphere. The determinant of the metric in the functional measure of integration over the fields $\pi(x)$ guarantees that the functional measure is a geometrical invariant under $O(N)$ induced transformations. If $Q$ denotes an element of $O(N)$, one can always define the matrix-valued function $Q(x)$ that relates $m(x)$ to $\hat{e}_1$ through

$$ g m(x) =: Q(x) \hat{e}_1, \quad g = +\sqrt{g^2} \geq 0, \quad (2.29) $$

The subgroup of $O(N)$ that leaves $\hat{e}_1$ invariant is called the little group (or stabilizer) of $\hat{e}_1$. Here, it is the subgroup $O(N - 1)$ of $O(N)$. If $R(x)$ takes values in the little group of $\hat{e}_1$,

$$ Q(x) R(x) \hat{e}_1 = Q(x) \hat{e}_1 = g m(x). \quad (2.30) $$

Relations (2.29) and (2.30) exhibit the isomorphism between the coset (homogeneous) space $O(N)/O(N - 1)$ and the $(N - 1)$–sphere $S_{N-1}$. More generally, Eq. (2.28b) can be taken as the definition of a NLSM on a $(N - 1)$–dimensional Riemannian manifold with local metric $g_{ij}$. This definition of a NLSM is more general than the $O(N)$–NLSM (2.14) as it is not always possible to establish an isomorphism between any given Riemannian manifold and some coset (homogeneous) space.

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$m$ is not smooth everywhere in $\mathbb{R}^d$, i.e., is singular at isolated points. On the lattice there is no notion of smoothness.

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A Riemannian manifold is a smooth manifold on which a continuous 2-covariant symmetric and nondegenerate tensor field called the metric tensor can be defined, i.e., for any point $x$ on the manifold there exists a symmetric and nondegenerate bilinear form $g_x$ from the tangent vector space at $x$ to the real numbers.
C. $O(N)$-NLSM as a field theory on a symmetric space

Representations (2.28a) and (2.28b) put the emphasis on the geometrical structure behind NLSM. The initial question on spontaneous-symmetry breaking is cast in the language of group theory. Is there a representation of the $O(N)$-NLSM that puts the emphasis on the underlying group theoretical structure, i.e., renders the pattern of symmetry breaking explicit?

On the one hand, Eqs. (2.28a) and (2.28b) say that, for any given $x \in \mathbb{R}^d$, $(N - 1)$ real parameters $\pi_1(x), \ldots, \pi_{N-1}(x)$, are needed to parametrize the $(N - 1)$-sphere.\(^6\) On the other hand, the number of independent generators of the coset space $O(N)/O(N - 1)$ is also $N - 1$.\(^7\)

This agreement is not coincidental as we saw in Eqs. (2.29) and (2.30). Indeed, we recall that for any point $g m(x)$ on the unit sphere $S_{N-1}$ with the co-ordinates $g \pi(x)$, there exists the $N \times N$ orthogonal matrix $Q(x) \in O(N)$ such that

$$g \pi(x) \sim g m(x) =: Q(x) \hat{e}_1.$$  \hspace{1cm} (2.35)

---

\(^6\) The $(N - 1)$-sphere is the $(N - 1)$-dimensional surface embedded in $\mathbb{R}^N$ and defined by

$$g^2 \left( \sigma^2 + \sum_{j=1}^{N-1} \pi_j^2 \right) = 1, \quad \forall \begin{pmatrix} \sigma \\ \pi \end{pmatrix} \in \mathbb{R}^N.$$  \hspace{1cm} (2.31)

The $(N - 1)$-sphere is often denoted $S_{N-1} \subset \mathbb{R}^N$.

\(^7\) For any $R \in O(N)$ det $R = \pm 1$. If det $R = 1$, i.e., $R \in SO(N)$, it is always possible to write $R = \exp(A)$ and $R^2 = \exp(A^2)$ where $A$ is also a $N \times N$ matrix with real-valued matrix elements. Equation (2.3) in footnote 1 implies that $A$ and $A^t$ obey

$$A + A^t = 0,$$  \hspace{1cm} (2.32)

i.e., that $A$ is a $N \times N$ real-valued antisymmetric matrix. The number of independent real-valued matrix elements in $A$ equals the number of entries above the diagonal, i.e.,

$$\frac{1}{2} (N^2 - N) = \frac{1}{2} N(N - 1).$$  \hspace{1cm} (2.33)

As real vector spaces, the dimensionality of $O(N)$ is thus $\frac{1}{2} N(N - 1)$ and the dimensionality of $O(N - 1)$ is $\frac{1}{2} (N - 1)(N - 2)$. The dimensionality of the coset space $O(N)/O(N - 1)$ is, by definition, the difference between the dimensionality of $O(N)$ and $O(N - 1)$,

$$\dim O(N)/O(N - 1) := \dim O(N) - \dim O(N - 1) = \frac{1}{2} (N - 1) [N - (N - 2)] = N - 1.$$  \hspace{1cm} (2.34)
Evidently, the relation between the point \( g \mathbf{m}(\mathbf{x}) \) of the unit sphere \( S_{N-1} \) and the \( N \times N \) rotation matrix \( R(\mathbf{x}) \) is one to many since right multiplication of \( Q(\mathbf{x}) \) by any element \( R(\mathbf{x}) \) from the little group \( O(N-1) \) that leaves the north pole \( \hat{\mathbf{e}}_1 \) unchanged yields
\[
Q(\mathbf{x}) \hat{\mathbf{e}}_1 = Q(\mathbf{x}) R(\mathbf{x}) \hat{\mathbf{e}}_1, \quad \forall R(\mathbf{x}) \in O(N-1).
\] (2.36)

The one-to-one relationship that we are seeking is between the unit sphere \( S_{N-1} \) and the quotient space of \( N \times N \) matrices \( O(N)/O(N-1) \) by which the co-ordinate \( g \pi(\mathbf{x}) \) of the point \( g \mathbf{m}(\mathbf{x}) \) on the unit sphere is represented by the set of matrices
\[
O(N)/O(N-1) := \{Q(\mathbf{x})R(\mathbf{x})|g \mathbf{m}(\mathbf{x}) = Q(\mathbf{x}) \hat{\mathbf{e}}_1 \text{ and } R(\mathbf{x}) \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_1\}. \quad (2.37)
\]

We are now going to represent the \( O(N) \)-NLSM in terms of the elements of \( O(N) \). To this end, observe that any real-valued \( N \times N \) antisymmetric matrix \( A(\mathbf{x}) \) can be written as
\[
A(\mathbf{x}) = \frac{g}{2} \sum_{1 \leq i < j \leq N} \alpha_{ij}(\mathbf{x}) T_{ij}, \quad T_{ij} := E_{ij} - E_{ji}, \quad (2.40)
\]
where the \( N \times N \) matrices \( E_{ij} \) has one single nonvanishing matrix element equal to 1 for line \( i \) and column \( j \), \( \alpha_{ij}(\mathbf{x}) \) are real-valued numbers, and \( \alpha_{ji} \) are smooth functions \( \mathbb{R}^d \to \mathbb{R} \). The factor of 1/2 is convention [see Eq. (2.39) in footnote 8] and we have endowed \( \alpha_{ij}(\mathbf{x}) \) with the dimensions of \( g^{-1} \). We shall assume that \( g \) is infinitesimal, in which case \( A \) is also infinitesimal. According to Eqs. (2.32) and (2.39) in footsteps 7 and 8, respectively, for an

\[
[T_{ij}, T_{kl}] = \delta_{ik}T_{lj} + \delta_{jl}T_{ki} + \delta_{il}T_{jk} + \delta_{jk}T_{il}, \quad 1 \leq i < j \leq N, \quad 1 \leq k < l \leq N, \quad (2.38)
\]
defines the Lie algebra of the Lie group \( SO(N) \). Since \( T_{ij} \) is antisymmetric with only two nonvanishing entries, +1 for line \( i \) and column \( j \) and −1 for line \( j \) and column \( i \),
\[
\text{tr} (T_{ij} T_{kl}) = \sum_{m,n=1}^{N} (T_{ij})_{mn} (T_{kl})_{nm} = \sum_{m,n=1}^{N} (\delta_{in} \delta_{jn} - \delta_{in} \delta_{jm}) (\delta_{km} \delta_{lm} - \delta_{km} \delta_{ln}) = 2 (\delta_{ij} \delta_{jk} - \delta_{ik} \delta_{jl}), \quad i,j,k,l = 1, \cdots, N.
\] (2.39)

The scaling factor of 1/2 in Eq. (2.40) insures that the trace of \( T_{ij}/2 \) with itself gives −1/2.
infinitesimal \( A = - A^t \)

\[
Q(x) = e^{A(x)} \approx I_N + A(x) \in SO(N),
\]

(2.41a)

\[
(Q^I \partial_\mu Q)(x) \approx \frac{g}{2} \sum_{1 \leq i < j \leq N} (\partial_\mu \alpha_{ji})(x) T_{ij},
\]

(2.41b)

\[
(Q^I \partial_\mu Q)^t(x) \equiv (Q \partial_\mu Q^t)(x) \approx - \frac{g}{2} \sum_{1 \leq i < j \leq N} (\partial_\mu \alpha_{ji})(x) T_{ij},
\]

(2.41c)

\[
\text{tr} \left[ (Q \partial_\mu Q^t)(x) (Q^I \partial_\mu Q)(x) \right] \approx \frac{g^2}{2} \sum_{1 \leq i < j \leq N} (\partial_\mu \alpha_{ji})^2(x).
\]

(2.41d)

The right-hand side of Eq. (2.41d) is positive and can thus be used to construct a Boltzman weight.

Next, we define the partition function

\[
Z_{g^2, H_0} := \left( \prod_{x \in \mathbb{R}^dO(N)} \int dQ(x) \right) e^{- \int d^d x \frac{g^2}{2} \text{tr} \left[ (Q^I D_\mu Q)^t (Q^I D_\mu Q) - H_0 I_{1,N-1} \right]},
\]

(2.42a)

\[
D_\mu Q := \partial_\mu Q - QA_\mu,
\]

(2.42b)

\[
A_\mu(x) := \text{Projection of } (Q^I \partial_\mu Q)(x) \text{ onto the little group.}
\]

(2.42c)

The integration measure for a connected matrix group is called the Haar measure. The measure of \( O(N) \) accounts for the fact that \( O(N) \) is not simply connected since it is impossible to connect the unit matrix \( I_N \) with the unit matrix \( -I_N \) in \( O(N) \) through a continuous path. In other words, one must sum separately over the two connected components of \( O(N) \), i.e., over those pure rotations with determinant +1, \( SO(N) \), and those rotation-inversions with determinant -1. The so-called covariant derivative \( D_\mu \) expresses the fact that out of the \( N(N-1)/2 \) degrees of freedom associated to the connected subgroup \( SO(N) \), \( (N-1)(N-2)/2 \) of them are redundant. Indeed, the “magnetic field” \( H_0 I_{1,N-1} \), here represented by the real number \( H_0 \) multiplying the diagonal matrix

\[
I_{1,N-1} := \begin{pmatrix}
+1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & -1
\end{pmatrix}
\]

(2.43)

in \( O(N) \), defines the subgroup \( O(N-1) \subset O(N) \) made of all matrices from \( O(N) \) that commute with \( I_{1,N-1} \). This is the little group. Any two local elements \( Q_1(x) \in SO(N) \) and
$Q_1(x) = Q_2(x) R(x)$, \hspace{1cm} (2.44)

are physically equivalent and should only be counted once in the path integral. To put it
differently, the covariant derivative insures that the path integral over $SO(N)$ reduces to
a path integral over all equivalences classes in the coset space $SO(N)/SO(N - 1)$. This
redundancy under local right multiplication is an example of a local gauge symmetry. The
transformation laws of the non-Abelian gauge field and covariant derivative under the right
multiplication

\begin{align}
Q(x) &\rightarrow Q(x) R(x), \quad R(x) \in SO(N - 1), \\
A_\mu &\rightarrow R^t A_\mu R + R^t \partial_\mu R \\
D_\mu Q &\rightarrow (D_\mu Q) R,
\end{align}

respectively. Hence, when $H_0 = 0$, the action in Eq. (2.42a) is locally gauge invariant due to
the cyclicity of the trace. Evidently, when $H_0 = 0$, the action in Eq. (2.42a) is also invariant
under the $O(N)$ global left multiplication

\begin{align}
Q(x) &\rightarrow L_0 Q(x) \\
\left(Q^t \partial_\mu Q\right)(x) &\rightarrow \left(Q^t \partial_\mu Q\right)(x) \\
\left(D_\mu Q\right)(x) &\rightarrow L_0 \left(D_\mu Q\right)(x).
\end{align}

When $H_0 \neq 0$, by the cyclicity of the trace, the global symmetry group is the transformation

\begin{align}
Q(x) &\rightarrow L_0^0 Q(x) L_0
\end{align}

where $L_0$ is any $N \times N$ matrix from the subgroup $O(N - 1)$ of matrices in $O(N)$ that
commute with $I_{1,N-1}$.

The partition function (2.42) shares the same global symmetries as the partition function (2.28b). Deriving the partition function (2.28b) from the partition function (2.42)
requires an explicit parametrization of the \( N \times N \) orthogonal matrices \( Q \). One possible choice can be found in chapter 6 from Ref. 1. However, the message of section II C that I am trying to convey is that a classical partition function can be interpreted as a gauge theory if a redundant description of the degrees of freedom is chosen.

I close this discussion with a sketch of some mathematical background.

An \( N \)-dimensional Riemannian manifold can be pictured as a smooth \( N \)-dimensional surface embedded in some Euclidean (flat) space through the imposition of a constraint. For example, the unit sphere \( S_{N-1} \) is the set of all \( N \)-dimensional real-valued vectors with unit length.

Riemannian manifolds are endowed with a metric, i.e., a notion of distance. For the case of the unit sphere \( S_{N-1} \) with the co-ordinates \( \pi_1, \ldots, \pi_{N-1} \), and the metric (2.23b), the distance between any two points follows from minimizing the length

\[
L[c] := g \int_0^1 dt \sqrt{g_{ij} \frac{d \pi_i}{dt} \frac{d \pi_j}{dt}}
\]

with respect to the choice made for the curve \( c(t) \) parametrized by \( 0 \leq t \leq 1 \) that connects the two points on the sphere. The minimal curve is called a geodesic.

The unit sphere \( S_{N-1} \) has, however, more than a metric. Any rotation of the Cartesian co-ordinate system in the embedding Euclidean space \( \mathbb{R}^N \) leaves the distance between any two points from the unit sphere unchanged. As a corollary, Eq. (2.35) holds.

This property of the unit sphere \( S_{N-1} \) can be generalized as follows. A Riemannian manifold \( \mathcal{M} \) is said to be homogeneous if it can be associated to a Lie group \( G \) in such a way that for any two point \( \mathbf{r} \) and \( \mathbf{\eta} \) in \( \mathcal{M} \)

- there exists an element from \( g \in G \) with \( g \mathbf{r} = \mathbf{\eta} \) (transitivity)

- and the distance between \( \mathbf{r} \) and \( \mathbf{\eta} \) is the same as the distance between \( g \mathbf{r} \) and \( g \mathbf{\eta} \) (isometry).

For example, the unit sphere \( S_{N-1} \) is an homogeneous Riemannian manifold with the transitive isometric group \( O(N) \).

An homogeneous Riemannian manifold \( \mathcal{M} \) is characterized by the coset \( G/H \) where \( H \) is the subgroup of \( G \) that leaves an arbitrary point \( \mathbf{r} \) of \( \mathcal{M} \) invariant,

\[
\mathbf{r} = h \mathbf{r}
\]
for any \( h \in H \). An homogeneous Riemannian manifold \( \mathfrak{M} \) is said to be symmetric if its symmetry group \( G \) (a semi-simple compact Lie group) is also characterized by a mapping on itself that preserves the group structure (an automorphism) and is involutive (it becomes the identity mapping if composed with itself). All elements of the Lie algebra of \( G \) are then either odd or even under this involution. The little group \( H \) is then generated by all the even generators from the Lie algebra of \( G \) under the involutive automorphism.

For example, a family of involutive automorphisms on \( O(N) \) are the mappings

\[
Q \to I_{p,q}^{-1} Q I_{p,q}
\]

where the \( N \times N \) diagonal matrices \( I_{p,q} \) are

\[
I_{p,q} = \text{diag}(+1, \cdots, +1, -1, \cdots, -1), \quad N = p + q.
\]

The family of subgroups of \( O(N) \) left invariant by these automorphisms is

\[
\{ O(p) \times O(N - p) \mid p = 1, \cdots, N - 1 \}.
\]

The corresponding family of coset spaces

\[
\{ \mathcal{G}_p \equiv O(N)/O(p) \times O(N - p) \mid p = 1, \cdots, N - 1 \},
\]

are called Grassmannian manifolds. The case \( p = 1 \) corresponds to the choice (2.43) that we made for the symmetry-breaking term in the partition function (2.42a). Thus, the \( O(N) \) NLSM is the special case when the target space is the \( p = 1 \) Grassmannian manifold \( \mathcal{G}_1 \). A one-to-one realization of \( \mathcal{G}_p \) in \( O(N) \) is given by

\[
x \to T(x) I_{p,N-p} T^{-1}(x), \quad T(x) \in O(N),
\]

since right multiplication

\[
T(x) \to T(x) R(x), \quad R(x) \in O(p) \times O(N - p) \subset O(N)
\]

leaves \( T(x) I_{p,N-p} T^{-1}(x) \) unchanged. The action

\[
S_{g^2, H_0} [Q] := \int \mathbb{R}^d d^4 x \frac{1}{g^2} \text{tr} \left[ (Q^\dagger \partial_\mu Q)^\dagger (Q^\dagger \partial_\mu Q) - H_0 I_{p,N-p} Q \right]
\]

where \( Q(x) \in O(N) \) is parametrized according to Eq. (2.57) delivers the Riemannian metric of \( \mathcal{G}_p \) once the trace has been evaluated (see Ref. 2).
D. Other examples of NLSM

1. Classical ferromagnetism with the group $O(3)$.

2. Liquid crystals.

3. Quantum antiferromagnets on a square lattice with the group $O(3)$.

4. Spin-1/2 quantum spin chains with the group $SU(2)$ which is locally isomorphic to $O(3)$.

5. Anderson localization, polymers, and other disordered systems.

6. Strongly correlated systems with the groups $SO(5)$ and $\mathbb{C}P_{N-1}$, the latter being locally isomorphic to $SU(N)/U(N-1)$.

III. FIXED POINT THEORIES, ENGINEERING AND SCALING DIMENSIONS, IRRELEVANT, MARGINAL, AND RELEVANT INTERACTIONS

For notational simplicity, I will consider the case $N = 1$ in this section. No fluctuations about the ferromagnetic state is allowed, irrespective of temperature, in the $O(1)$–NLSM,

$$1 = \int \mathcal{D}[\phi] \delta \left( \frac{1}{g^2} - \phi^2 \right) \exp \left( -\frac{1}{2} \int_{\mathbb{R}^d} d^d x \left[ \left( \partial_\mu \phi \right)^2 - 2 h \phi \right] \right) = Z_{\beta, h}. \quad (3.1)$$

Instead of the $O(1)$–NLSM, consider the interacting field theory for the scalar field $\varphi$ defined by

$$Z = \int \mathcal{D}[\varphi] \exp \left( - \left( S_0 + S_1 \right) \right), \quad (3.2a)$$

$$S_0 = \int_{\mathbb{R}^d} d^d x \mathcal{L}_0, \quad \mathcal{L}_0 = \frac{1}{2} \left( \partial_\mu \varphi \right)^2, \quad (3.2b)$$

$$S_1 = \int_{\mathbb{R}^d} d^d x \mathcal{L}_1, \quad \mathcal{L}_1 = V(\varphi). \quad (3.2c)$$

As usual $\hbar = c = 1$, and the action

$$S = S_0 + S_1 \quad (3.3)$$
must be dimensionless, sitting as it is in the argument of an exponential. Consequently, the
Lagrangian density
\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \] (3.4)
has dimension
\[ [\mathcal{L}] = (\text{length})^{-d}. \] (3.5)
By convention (stemming from high-energy physics whereby dimensions are counted in in-
verse powers of length, i.e., in powers of momentum) the engineering dimension of \( \mathcal{L} \) is \( d \).
The engineering dimension of the scalar field can be read from the kinetic energy \( \mathcal{L}_0 \),
\[ [\varphi] = (\text{length})^{-(d-2)/2}. \] (3.6)
Thus, \( \varphi \) has engineering dimension \( (d-2)/2 \). In particular, the engineering dimension of
the scalar field is:
1. \(-1/2\) if \( d = 1 \).
2. \(0\) if \( d = 2 \).
3. \(+1/2\) if \( d = 3 \).
4. \(1\) if \( d = 4 \).

When the interaction potential vanishes,
\[ V(\varphi) = 0, \] (3.7)
any rescaling of space
\[ x = \kappa \tilde{x} \iff a = \kappa \tilde{a}, \quad 0 < \kappa < \infty, \] (3.8)
(\( a \) is the initial microscopic length scale, say the lattice spacing, \( \tilde{a} \) is the rescaled microscopic
length scale) can be compensated by the rescaling
\[ \varphi = \kappa^{-(d-2)/2} \tilde{\varphi} \] (3.9)
of the scalar field \( \varphi \) so as to insure the invariance of the action \( S = S_0 \) under this rescaling,
\[ S = \int_{\mathbb{R}^d} d^d x \frac{1}{2} (\partial_\mu \varphi)^2 = \int_{\mathbb{R}^d} d^d \tilde{x} \frac{1}{2} (\partial_\mu \tilde{\varphi})^2 = \tilde{S}. \] (3.10)
Equation (3.10) encodes the property of *scale invariance*. The action of the free scalar field theory is scale invariant. The partition function $Z = Z_0$ of the free scalar field theory is not scale invariant since Eq. (3.9) changes the partition function by an infinite multiplicative factor (the factor $\kappa^{-(d-2)/2}$ for each $x$). However, this infinite multiplicative factor drops out of all correlation functions

$$
\langle \varphi(x_1) \cdots \varphi(x_m) \varphi(y_1) \cdots \varphi(y_n) \rangle_{Z_0} = \left[ \frac{\mathcal{D}[\varphi]}{\mathcal{D}[\varphi]} \right] e^{-S_0} - \frac{\partial^{m+n} Z_{0,J}}{\partial J(x_1) \cdots \partial J(y_n)} \bigg|_{J=0},
$$

(3.11)

Scale invariance of the action $S_0$ fixes the engineering dimension of $(m+n)$-point correlation functions of the free scalar field,

$$
\left[ \langle \varphi(x_1) \cdots \varphi(x_m) \varphi(y_1) \cdots \varphi(y_n) \rangle \right]_{Z_0} = \text{(length)}^{-(d-2)(m+n)/2}.
$$

(3.12)

Equation (3.12) is tantamount to guessing that, up to some dimensionless multiplicative prefactor,

$$
\langle \varphi(x) \varphi(y) \rangle_{Z_0} \propto \left( \frac{1}{|x-y|^2} \right)^{(d-2)/2}, \quad d \in \mathbb{N} \setminus \{2\}.
$$

(3.13)

This guess is confirmed by direct computation of the Fourier transform of the free scalar field propagator $1/k^2$ in momentum space,

$$
D(x) := \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} e^{ik \cdot x} = \frac{\Gamma \left( \frac{d-2}{2} \right)}{4\pi^{d/2}} \left( \frac{1}{|x|^2} \right)^{(d-2)/2}, \quad d \in \mathbb{N} \setminus \{2\}.
$$

(3.14)

The case of two space-time dimension, $d = 2$, is very special in that $\varphi$ is itself scale invariant. This is reflected by the singularity of the gamma function at the origin.\(^9\) I will devote a subsection to this case below.

\(^9\) The gamma function has the integral representation

$$
\Gamma(z) := \int_0^\infty dt \frac{e^{-t}}{t^{z-1}}, \quad z \in \mathbb{C}.
$$

(3.15)

The gamma function is single-valued and analytic over the entire complex plane, save for the points $z = 0, -1, -2, -3, \cdots$ where it possesses single poles with residues $(-1)^n/n!$. 


Predictions from scale invariance of $S_0$ can be circumvented by other symmetries of $S_0$. For example, $S_0$ is also invariant under the discrete symmetry

$$\varphi = -\tilde{\varphi}.$$  

(3.16)

This symmetry implies that the $(m+n)$-point correlation function (3.12) vanishes whenever $m+n$ is odd.

Scale invariance of $S_0$ has a limited predictive power for the spatial dependence of $2m$-point correlation function, $m > 1$. One must rely on a direct calculation to show that the $2m$-point correlation function reduces to a sum over $m$ products of two-point functions. This result is known as the Wick theorem in an operator representation of quantum field theory. With path integral techniques this result simply follows from application of the product rule for differentiation.  

A. Fixed-point theories

Consider the family of partition function $\{Z\}_V$ labeled by the interacting potential $V(\varphi)$ in Eq. (3.2c). A fixed-point theory $Z^*$ from this family of theories has an action $S^* = S_0 + S_1^*$ that is scale invariant under simultaneous rescaling of space-time $x$ and $\varphi$. We have already encountered one fixed-point theory, the free-field-fixed-point theory when the interaction potential $V(\varphi)$ vanishes. One could imagine that there are other potentials for which scale invariance is realized. At a fixed point, scale invariance dictates that $(m+n)$-point correlation functions [Eq. (3.11) with $S_0 \rightarrow S_0 + S_1^*$] are algebraic functions in any dimensions other than $d = 2$. The 2-point function can then be used to define the scaling dimension $\delta_\varphi$ of the scalar field at a fixed point,

$$\langle \varphi(x)\varphi(y) \rangle_{Z^*} \propto a^{-(d-2)} \left( \frac{a^2}{|x-y|^2} \right)^{\delta_\varphi}, \quad d \in \mathbb{N} \setminus \{2\}. \quad (3.17)$$

10 Correlation functions in a field field theory are obtained from the partition function $Z_J$ in the presence of a source field $J$ that couples linearly to the free fields. For a free field theory, the generating function $Z_J$ is proportional to $\exp(+JGJ/2)$ where $G$ is the free-field Green function since the path integral is Gaussian. Repeated differentiation with respect to the source field of the generating function yields all correlation function once the limit $J \rightarrow 0$ is taken. Wick theorem is just an application of the product rule to the $n$-th order differentiation of $\exp(+JGJ/2)$ with respect to $J$. 

22
The engineering dimension of the correlation function is made explicit by the introduction of the microscopic length scale \( a \), say the lattice spacing. The proportionality constant is some dimensionless numerical factor. The free-field-fixed-point theory is characterized by the fact that engineering and scaling dimensions coincide. This need not be true anymore at some putative interacting fixed-point theory where \( V^*(\varphi) \neq 0 \).

The physical significance of a fixed-point theory depends on the way any perturbation to the fixed-point theory behaves under rescaling. Consider for example the perturbation

\[
V_m(\varphi) := \frac{1}{2m} \lambda_m \varphi^{2m}, \quad m \in \mathbb{N}, \quad 0 < \lambda_m \in \mathbb{R},
\]

(3.18)

to the free-field fixed point theory \( Z^* = Z_0 \). At the free-field fixed point, we need not distinguish engineering from scaling dimensions. The dimension of the coupling constant \( \lambda_m \) is

\[
[\lambda_m] = (\text{length})^{-d+(d-2)m}
\]

(3.19)
since

\[
S_1 = \int_{\mathbb{R}^d} d^d x \frac{1}{2m} \lambda_m \varphi^{2m}
\]

(3.20)
is dimensionless and the scaling dimension of \( \varphi \) is fixed by Eq. (3.6). Thus, under length rescaling (3.8),

\[
\lambda_m = \kappa^{-d+(d-2)m} \tilde{\lambda}_m.
\]

(3.21)

Choose the rescaling factor \( 0 < \kappa < 1 \). The rescaled coupling constant

\[
\tilde{\lambda}_m = \kappa^{d-(d-2)m} \lambda_m, \quad 0 < \kappa < 1,
\]

(3.22)

- is smaller than the original one if

\[
(d - 2)m < d,
\]

(3.23)
- is unchanged if

\[
(d - 2)m = d,
\]

(3.24)
- is larger than the original one if

\[
(d - 2)m > d.
\]

(3.25)
Correspondingly, the interaction $V_m$ is said to be

- **UV irrelevant if**
  \[(d - 2)m < d,\]  \((3.26)\)

- **marginal if**
  \[(d - 2)m = d,\]  \((3.27)\)

- **UV relevant if**
  \[(d - 2)m > d.\]  \((3.28)\)

The terminology of irrelevance and relevance depends on the choice between $0 < \kappa < 1$ or $1 < \kappa < \infty$. Irrelevant interactions when $0 < \kappa < 1$ become relevant interactions when $1 < \kappa < \infty$ and vice versa. The choice $0 < \kappa < 1$ consists in zooming into the microscopic scale in that the new microscopic length scale $\tilde{a} = a/\kappa$ by which all lengths are measured now appears larger than the original one $a$. The choice $0 < \kappa < 1$ is made if one is interested in the asymptotic behavior of correlation functions at short distances or short wave lengths. This is the ultra-violet (UV) limit of primary interest in high-energy physics. The choice $1 < \kappa < \infty$ consists in zooming away from the microscopic scale in that the new microscopic length scale $\tilde{a} = a/\kappa$ by which all lengths are measured now appears smaller than the original one $a$. The choice $1 < \kappa < \infty$ is made if one is interested in the asymptotic behavior of correlation functions at long distances or long wave lengths. This is the infra-red (IR) limit of primary interest in condensed matter physics.\(^{11}\)

The rescaled coupling constant

\[\tilde{\lambda}_m = \kappa^{d-(d-2)m} \lambda_m, \quad 1 < \kappa < \infty,\]  \((3.32)\)

\(^{11}\) Assume that the short-distance cutoff is $a - da$ to begin with. Imagine that one integrates over all length scales between $a - da$ and $a$, say by breaking up integrals into

\[\int_{a-\text{da}}^{\infty} dr \cdots = \int_{a-\text{da}}^{a} dr \cdots + \int_{a}^{\infty} dr \cdots.\]  \((3.29)\)

Integration over the interval $[a - da, a]$ can sometimes be absorbed into a redefinition of the coupling constants of the theory. If so, one is left with

\[\int_{a}^{\infty} dr \cdots = \int_{a-\text{da}}^{\infty} d\tilde{r} \cdots.\]  \((3.30)\)
• is larger than the original one if

\[(d - 2)m < d,\]  
(3.33)

• is unchanged if

\[(d - 2)m = d,\]  
(3.34)

• is smaller than the original one if

\[(d - 2)m > d.\]  
(3.35)

Correspondingly, the interaction \(V_m\) is said to be

• IR relevant if

\[(d - 2)m < d,\]  
(3.36)

• marginal if

\[(d - 2)m = d,\]  
(3.37)

• IR irrelevant if

\[(d - 2)m > d.\]  
(3.38)

Observe that a mass term is relevant in any dimensions in the IR limit [(Eq. (3.36) with \(m = 1\)].

At a generic IR fixed-point, it is the scaling dimension \(\delta_O\) of a field \(O\), not the engineering dimension \(-\log[O]\) in units of length, that decides of the relevance, marginality, or irrelevance of the “small perturbation” \(O\), whereby it is imagined that

\[S_O = \int_{\mathbb{R}^d} d^d x \ \lambda_O \ O, \quad 0 \leq a^{-d+\log[O]}\lambda_O \ll 1,\]  
(3.39)

Here, form invariance has been restored on the right-hand side with the help of the rescaling [compare with Eq. (3.8)]

\[r = \frac{a}{a - da \tilde{r}}.\]  
(3.31)
has been added to the fixed-point action $S^*$. Under rescaling $a = \kappa \tilde{a}$,
\begin{equation}
S_O = \int_{\mathbb{R}^d} d^d \vec{x} \, \kappa^{d-\delta_O} \lambda_O \, \tilde{O},
\end{equation}
and the perturbation $O$ is said to be

- **IR relevant if**
  \begin{equation}
  \delta_O < d,
  \end{equation}

- **marginal if**
  \begin{equation}
  \delta_O = d,
  \end{equation}

- **IR irrelevant if**
  \begin{equation}
  \delta_O > d.
  \end{equation}

**B. Two-dimensional $O(2)$–NLSM in the spin-wave approximation**

To illustrate the peculiarities of two-dimensional space-time, consider the $O(2)$–NLSM in $d = 2$ with the partition function
\begin{equation}
Z_{\beta, \mathcal{H}} := \int \mathcal{D}[\mathbf{n}] \delta \left( 1 - n^2 \right) \exp \left( -\frac{1}{2g^2} \int_{\mathbb{R}^2} d^2 x \left[ (\partial_\mu \mathbf{n})^2 - 2a^{-2} \beta g^2 \mathbf{H} \cdot \mathbf{n} \right] \right).
\end{equation}
The surface of the unit sphere in $\mathbb{R}^2$, the one sphere, is simply the unit circle. Planar spins of unit length can be represented as complex numbers of unit length, i.e., phases,
\begin{equation}
\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad n_1 = \cos \phi = \text{Re} \varphi, \quad n_2 = \sin \phi = \text{Im} \varphi, \quad \varphi = e^{i\phi}.
\end{equation}
Hence,
\begin{equation}
Z_{\beta, \mathcal{H}} \propto \int \mathcal{D}[\varphi^*, \varphi] \delta \left( 1 - |\varphi|^2 \right) e^{-\frac{1}{2g^2} \int_{\mathbb{R}^2} d^2 x \left[ |\partial_\mu \varphi|^2 - 2a^{-2} \beta g^2 \left( H_1 \varphi^* \varphi + H_2 \varphi^* \varphi^* \right) \right]}.
\end{equation}

\footnote{The proportionality constant results from the change of the normalization of the measure in the path integral,
\begin{equation}
\frac{dn_1(x)dn_2(x)}{\pi} \rightarrow \frac{d\varphi^*(x)d\varphi(x)}{2\pi^1}.
\end{equation}
In this way a Gaussian integral is normalized to the inverse of the determinant of the kernel.}
Without loss of generality, spontaneous-symmetry breaking into the ferromagnetic ground state is enforced by taking the external magnetic field to be \( \mathbf{H} = |\mathbf{H}| \hat{e}_1 \) and letting \( |\mathbf{H}| \to 0 \) at the end of the day. Hence, the ferromagnetic state is

\[
\varphi(x) = 1, \quad \forall x \in \mathbb{R}^2. \tag{3.48}
\]

At finite temperature, the path-integral representation of the partition function will be restricted to small deviations \( \varphi(x) \) about the ferromagnetic state (3.48). To be more precise, the spin-wave approximation by which the angular field \( \phi(x) = \arg[\varphi(x)] \) is rotation free,

\[
0 = \oint_x d\mathbf{x}_\mu \epsilon_{\mu\nu} \partial_\nu \phi, \quad \forall x \in \mathbb{R}^2, \tag{3.49a}
\]

is made. Here, \( \oint_x \) denotes any closed line integral that encloses \( x \) and

\[
\epsilon_{12} = -\epsilon_{21} = 1, \quad \epsilon_{11} = \epsilon_{22} = 0, \tag{3.49b}
\]
i.e., \( \phi \) is smooth and single-valued everywhere. Inclusion in the path integral of multi-valued configurations of \( \phi \) leads to the so-called Kosterlitz-Thouless transition. However, I will ignore this important aspect of the problem as my goal is to illustrate how a perturbative RG procedure can be performed on the \( O(N) \) NLSM whereas the physics of the Kosterlitz-Thouless transition is nonperturbative (with respect to \( g^2 \)) by nature.

I want to compute the \((m + n)\)-point correlation function

\[
G_{\text{sw}, g^2, H=0}^{(m,n)}(x_1, \ldots, x_m, y_1, \ldots, y_n) := \langle \varphi(x_1) \cdots \varphi(x_m) \varphi^*(y_1) \cdots \varphi^*(y_n) \rangle_{\text{sw}, g^2, H=0} \\
= \langle e^{+i\phi(x_1)} \cdots e^{+i\phi(x_m)} e^{-i\phi(y_1)} \cdots e^{-i\phi(y_n)} \rangle_{\text{sw}, g^2, H=0}, \tag{3.50a}
\]

within the spin-wave approximation of the \( O(2) \)-NLSM in \( 2d \), i.e., with the partition function

\[
Z_{\text{sw}, \beta, H} \propto \int \mathcal{D}[\phi] e^{-\frac{1}{g^2} \int d^2x \left[ (\partial_\mu \phi)^2 - 2\epsilon_{22} g^2 (H_1 \cos \phi + H_2 \sin \phi) \right]}. \tag{3.50b}
\]

Observe that the partition function (3.50b) is unchanged under

\[
\phi = \tilde{\phi} + \text{const} \tag{3.51}
\]
in the thermodynamic limit and when \( \mathbf{H} \) vanishes. This immediately implies that the correlation function (3.50a) vanishes unless

\[
m = n. \tag{3.52}
\]
With a vanishing external magnetic field, the identity
\[
\left\langle e^{+i \int d^2 x J(x) \phi(x)} \right\rangle_{\text{sw } g^2, H=0} = e^{-\frac{\mu^2}{4g^2} \int d^2 x \int d^2 y J(x) G(x,y) J(y)}
\] (3.53a)
holds for any source \( J(x) \). Here, \( G(x, y) \) is the Green function defined by
\[
\left( -\frac{1}{g^2} \partial^2 \right) G(x, y) = \delta(x - y).
\] (3.53b)

In other words,
\[
G(x, y) = \lim_{M^2 \to 0} g^2 \int \frac{d^2 k}{(2\pi)^2} \frac{e^{ik \cdot (x-y)}}{k^2 + M^2}
= \lim_{M^2 \to 0} g^2 \left[ -\frac{1}{2\pi} \ln (M|x - y|) + \text{const} + O(M|x - y|) \right].
\] (3.54)

Strictly speaking, the Green function is ill-defined because of the logarithmic singularities in the infra-red limit \( M \to 0 \) and in the ultra-violet limit \( |x - y| \to 0 \). The ultra-violet singularity can be removed by introducing a high-energy cut-off, say the inverse lattice spacing \( 1/a \). The infra-red cut-off \( M \) then drops out of the difference
\[
G(x, y) - G(x, x + a \hat{r}) = -\frac{g^2}{2\pi} \ln \left( \frac{|x - y|}{a} \right) + O(M|x - y|)
\] (3.55)
(\( \hat{r} \) is some unit length vector) to leading order in \( |x - y|/a \). To compute the correlation function (3.50a) it suffices to choose the source
\[
J(x) := \sum_{i=1}^m \delta(x - x_i) - \sum_{j=1}^n \delta(x - y_j)
\] (3.56)
in Eq. (3.53a) which gives
\[
G_{\text{sw } g^2, H=0}(x_1, \ldots, x_m; y_1, \ldots, y_n) =
\left\langle e^{+i \int d^2 x J(x) \phi(x)} \right\rangle_{\text{sw } g^2, H=0}
= e^{-\frac{\mu^2}{4g^2} \sum_{i,j=1}^m G(x_i - x_j)} \times e^{+\frac{\mu^2}{2g^2} \sum_{k,l=1}^n G(y_k - y_l)} \times e^{+2\frac{\mu^2}{g^2} \sum_{i=1}^m \sum_{l=1}^n G(x_i - y_l)}
= (Ma)^{+\frac{2}{4\pi}(m+n)} \prod_{1 \leq i \neq j \leq m} (M|x_i - x_j|) ^{+\frac{2}{4\pi}} \prod_{1 \leq k \neq l \leq n} (M|y_k - y_l|) ^{+\frac{2}{4\pi}} \prod_{i=1}^m \prod_{l=1}^n (M|x_i - y_l|) ^{+\frac{2}{4\pi}}
= M^{+\frac{2}{4\pi}(m-n)^2} a^{+\frac{2}{4\pi}(m+n)} \left[ \prod_{1 \leq i < j \leq m} |x_i - x_j| \right]^{+\frac{2}{4\pi}} \left[ \prod_{1 \leq k < l \leq n} |y_k - y_l| \right]^{+\frac{2}{4\pi}} \left[ \prod_{i=1}^m \prod_{l=1}^n |x_i - y_l| \right]^{+\frac{2}{4\pi}}
\] (3.57)
This expression remains well defined when the infra-red cut-off is removed, \( M \to 0 \), as long as the short distance cut-off \( a \) is kept finite, in which case

\[
G_{sw, H=0}^{(m,n)}(x_1, \ldots, x_m, y_1, \ldots, y_n) = \begin{cases} 
0, & \text{if } m \neq n, \\
 a^{\frac{g^2}{2\pi}} \left[ \prod_{1 \leq i < j \leq m} |x_i - x_j| \right]^{\frac{1}{2}} \left[ \prod_{1 \leq i < j \leq m} |y_i - y_j| \right]^{\frac{1}{2}} + \frac{g^2}{2\pi}, & \text{if } m = n. 
\end{cases} \quad (3.58)
\]

The case \( m = n = 1 \) gives the two-point function

\[
G_{sw, H=0}^{(1,1)}(x, y) = \left( \frac{a}{|x - y|} \right) + \frac{g^2}{2\pi}. \quad (3.59)
\]

All correlation functions of the form (3.50a) are thus algebraic functions for any given finite value of \( g^2 \). At zero temperature, i.e., when \( g^2 = 0 \), all correlation functions are constant as it should be if the ground state supports ferromagnetic long-range order (LRO). Within the spin-wave approximation, LRO at zero temperature \( (g^2 = 0) \) is downgraded to algebraic order or quasi-long-range order (QLRO) at any finite temperature. Equation (3.57) defines a critical phase of matter for any given \( g^2 \). At criticality scale invariance manifests itself by algebraic decaying correlation functions. Here, the critical phase of matter is called the spin-wave phase. Direct inspection of the two-point function (3.59) allows us to infer that the scaling dimension \( \delta_{\varphi} \) of the field \( \varphi \) is given by

\[
\delta_{\varphi} = \frac{g^2}{4\pi}. \quad (3.60)
\]

This scaling dimension is a smooth function of \( g^2 \) and is different from the engineering dimension of \( \varphi \) which is zero.

Correlation functions in the spin-wave phase are ambiguous in the limit \( a \to 0 \) in which the ultra-violet cut-off is removed. This ambiguity can be interpreted as follows. The accuracy of the spin-wave approximation improves at low energies and long distances, i.e., scaling exponents controlling the algebraic decay of correlation functions can be thought of as being exact or, more precisely, universal in that they do not depend on the prescription used to regularize the theory at short distances. Short distance regularizations in condensed matter physics are much more than a mathematical artifact as they refer to a specific lattice or microscopic model. The mathematical ambiguity in the choice of an ultra-violet
cut-off reflects the property that lattice models that differ on the microscopic scale might nevertheless share the same properties at low energies and long distances. From the point of view of physics this is a very important property called \textit{universality} without which the task of classifying and predicting phases of condensed matter would otherwise be hopeless.

The mathematical ambiguity in the choice of an ultra-violet cut-off can be encoded in a differential equation obeyed by correlation functions. This differential equation is called the \textit{Callan-Symanzik} equation. The construction of the Callan-Symanzik equation in the spin-wave phase proceeds as follows. The ambiguity in the choice of the ultra-violet cut-off $1/a$ can be quantified by introducing a \textit{renormalization point} or \textit{renormalization mass} $\mu$

\begin{equation}
\left( \prod_{1 \leq i < j \leq m} |x_i - x_j| \right) \left( \prod_{1 \leq i < j \leq m} |y_k - y_l| \right) \frac{\prod_{i,j=1}^m |x_i - y_l|}{\prod_{i,j=1}^m |x_i - y_j|} = (a\mu)^{\frac{2}{\pi}} \left( \frac{1}{\mu |x - y|} \right)^{\frac{2}{\pi}} \left( \prod_{1 \leq i < j \leq m} \mu |x_i - x_j| \right) \left( \prod_{1 \leq i < j \leq m} \mu |y_k - y_l| \right) \frac{\prod_{i,j=1}^m \mu |x_i - y_l|}{\prod_{i,j=1}^m \mu |x_i - y_j|} \right)^{\frac{2}{\pi}} (3.61)
\end{equation}

for the $(2m)$-point function (3.58) and

\begin{equation}
\left( \frac{a}{|x - y|} \right)^{\frac{2}{\pi}} = (a\mu)^{\frac{2}{\pi}} \left( \frac{1}{\mu |x - y|} \right)^{\frac{2}{\pi}} (3.62)
\end{equation}

for the 2-point function (3.59). Define the \textit{renormalized field}

\begin{equation}
\phi^{(R)} := \frac{1}{\sqrt{Z}} \phi, (3.63)
\end{equation}

whereby

\begin{equation}
\sqrt{Z} := (a\mu)^{\frac{2}{\pi}} = e^{\frac{2}{\pi} \ln(a\mu)}. (3.64)
\end{equation}

The original field $\phi$ is called the \textit{bare} or \textit{unrenormalized} field. The dimensionless number $Z$ is called the \textit{wavefunction renormalization factor}. Correlation functions (3.57) and (3.59)
can be expressed in terms of the renormalized fields as

\[ G^{(m,m)}(R)_{swg^2,H=0}(x_1, \ldots, x_m, y_1, \ldots, y_m) = \langle \varphi^{(R)}(x_1) \cdots \varphi^{(R)}(x_m) \varphi^{(R)*}(y_1) \cdots \varphi^{(R)*}(y_m) \rangle_{swg^2,H=0} \]

\[ = \left[ \frac{\prod_{1 \leq i < j \leq m} \mu |x_i - x_j|}{\prod_{i,j=1}^m \mu |x_i - y_j|} \right]^{1/2} \left[ \frac{\prod_{1 \leq i < j \leq m} \mu |y_k - y_l|}{\prod_{i,j=1}^m \mu |x_i - y_l|} \right]^{1/2} \]

(3.65)

and

\[ \langle \varphi^{(R)}(x) \varphi^{(R)*}(y) \rangle_{swg^2,H=0} = \left( \frac{1}{\mu |x - y|} \right)^{1/2}, \]

(3.66)

respectively. We have thus traded the ultra-violet cut-off $1/a$ for $\mu$. The Callan-Symanzik equation obeyed by the correlation function (3.58) follows from the observation that Eq. (3.64) does not depend on $\mu$,

\[ 0 = \mu \frac{d}{d\mu} G^{(m,m)}_{swg^2,H=0}(x_1, \ldots, x_m, y_1, \ldots, y_m) \]

Eqs. (3.65) and (3.63) \[ = \mu \frac{d}{d\mu} \left[ Z^m G^{(m,m)}_{swg^2,H=0}(x_1, \ldots, x_m, y_1, \ldots, y_m) \right] \]

\[ = Z^m \left( \mu \frac{\partial}{\partial \mu} + 2m \frac{\partial \ln \sqrt{Z}}{\partial \mu} \right) G^{(m,m)}_{swg^2,H=0}(x_1, \ldots, x_m, y_1, \ldots, y_m) \]

Eq. (3.64) \[ = Z^m \left( \mu \frac{\partial}{\partial \mu} + 2m \frac{g^2}{4\pi} \right) G^{(m,m)}_{swg^2,H=0}(x_1, \ldots, x_m, y_1, \ldots, y_m) \]

\[ \equiv Z^m \left[ \mu \frac{\partial}{\partial \mu} + 2m \gamma(g^2) \right] G^{(m,m)}_{swg^2,H=0}(x_1, \ldots, x_m, y_1, \ldots, y_m). \]

(3.67)

The anomalous scaling dimension

\[ \gamma(g^2) := \mu \frac{\partial \ln \sqrt{Z}}{\partial \mu} = \frac{g^2}{4\pi} \]

(3.68)

has been introduced. It is the difference between the scaling and the engineering dimension of the field $\varphi$.

The lessons learned from the example of the 2d $O(2)$-NLSM are:
1. The vanishing of the $m \neq n$ correlator (3.50a) as the infra-red cut-off $M \to 0$ guaranties that the $O(2)$ symmetry is not spontaneously broken at any finite temperature. This is an example of the Hohenberg-Mermin-Wagner theorem that asserts that no continuous global symmetry can be spontaneously broken in $d \leq 2$.

2. Correlator (3.50a) depends on an ultra-violet cut-off. This dependence can be quantified by the Callan-Symanzik equation obeyed by renormalized fields.

3. Anomalous scaling dimensions that appear in the Callan-Symanzik equation are universal in that they are independent of the choice of the ultra-violet cut-off.

4. The spin-wave phase is a critical or QLRO phase in which correlator (3.50a) is an algebraic decaying functions at any finite temperature. Anomalous scaling dimensions are continuous functions of the temperature.

IV. GENERAL METHOD OF RENORMALIZATION

In this section I am going to set up the Callan-Symanzik equation obeyed by correlation functions in all generality. Consider some bare correlation function

$$G^{(m,n)}_B(z; g_B, \Lambda)$$

between $(m + n)$ local fields. Here, $z$ denotes collectively the $(m + n)$ space arguments of the local fields,

$$z = \{x_1, \ldots, x_m, y_1, \ldots, y_n\},$$

and $g_B$ denotes collectively all (IR) relevant coupling constants at the free field fixed point,

$$g_B = \{g^{(1)}_B, g^{(2)}_B, \ldots\}.$$  

It is commonly assumed that the number of relevant coupling constants is finite but this need not be so, for example when dealing with disordered systems. The inverse of the lattice spacing

$$\Lambda = \frac{1}{a}$$
is taken as the UV cut-off. I now assume that it is possible to express the correlator (4.1a) in terms of a wave-function renormalization factor $Z$, renormalized coupling constants $g_R$, and a new renormalization point $\mu$ according to

$$G_R^{(m,n)}(z; g_R, \mu) = [Z(g_R, \mu/\Lambda)]^{\frac{m+n}{2}} \times G_R^{(m,n)}(z; g_R, \mu). \quad (4.3)$$

Equation (4.3) is certainly not correct when a pair of spatial arguments of the correlator is within a distance of the order of the lattice spacing $a$ as the renormalized correlator must then also depend on $\Lambda$. However, Eq. (4.3) becomes plausible when all spatial arguments are separated pairwise by an amount much larger than the lattice spacing $a$. In any case Eq. (4.3) is to be verified by explicit computation as we did for the spin-wave phase of the 2d $O(2)$ NLSM. By assumption Eq. (4.3) implies the Callan-Symanzik equation

$$0 = \left[\mu \partial_\mu + \beta(g_R) \partial_{g_R} + (m+n)\gamma(g_R)\right] G_R^{(m,n)}(z; g_R, \mu), \quad (4.4a)$$

$$\beta(g_R) := \mu \partial_\mu g_R \quad \text{at fixed } g_B \text{ and } \Lambda, \quad (4.4b)$$

$$\gamma(g_R) := \mu \partial_\mu \ln \sqrt{Z} \quad \text{at fixed } g_B \text{ and } \Lambda. \quad (4.4c)$$

Observe that we could have equally well written

$$G_B^{(m,n)}(z; g_B, \Lambda) = Z^{-\frac{m+n}{2}}(g_B, \mu/\Lambda) \times G_B^{(m,n)}(z; g_B, \Lambda) \quad (4.5)$$

instead of Eq. (4.3) to derive the Callan-Symanzik equation

$$0 = \left[\Lambda \partial_\Lambda + \tilde{\beta}(g_B) \partial_{g_B} - (m+n)\tilde{\gamma}(g_B)\right] G_B^{(m,n)}(z; g_B, \Lambda), \quad (4.6a)$$

$$\tilde{\beta}(g_B) := \Lambda \partial_\Lambda g_B \quad \text{at fixed } g_R \text{ and } \mu, \quad (4.6b)$$

$$\tilde{\gamma}(g_B) := \Lambda \partial_\Lambda \ln \sqrt{Z} \quad \text{at fixed } g_R \text{ and } \mu. \quad (4.6c)$$

The function $\beta(g_R) [\tilde{\beta}(g_B)]$ quantifies the rate of change of the renormalized (bare) coupling constants as the renormalization point (lattice spacing) is varied. The flow of the coupling constants under an infinitesimal change in the renormalization point (lattice spacing) is thus controlled by the so-called beta function. For the 2d $O(2)$ NLSM in the spin-wave approximation there is only one coupling constant $g^2$ that does not flow, i.e., the beta function of $g^2$ vanishes identically as it should be at a critical point.
V. PERTURBATIVE EXPANSION OF THE TWO-POINT CORRELATION FUNCTION UP TO ONE LOOP FOR THE TWO-DIMENSIONAL $O(N)$ NLSM

I am going to compute the two-point correlator for the field $n$ entering the $2d\,O(N > 2)$ NLSM (2.14) through a perturbative expansion in powers of $g^2$ up to order $g^4$ and within the spin-wave approximation (2.28a). I am thus after the expansion of

$$G_{sw}^{(1,1)}(x, y; a) := \langle n(x) \cdot n(y) \rangle_{sw;a}$$

$$= g^2 \langle m(x) \cdot m(y) \rangle_{sw;a}$$

$$= g^2 \left( \sqrt{\frac{1}{g^2 - \pi^2(x)}} \sqrt{\frac{1}{g^2 - \pi^2(y)}} \right)_{sw;a} + g^2 \langle \pi(x) \cdot \pi(y) \rangle_{sw;a}$$

up to order $g^4$, where the expectation value $\langle (\cdots) \rangle_{sw;a}$ is defined by

$$\langle (\cdots) \rangle_{sw;a} := \int D[\pi] \Theta(1 - g^2\pi^2)e^{-\frac{1}{2} \int_{\mathbb{R}^2} d^2x (\partial_i \pi_i) \left[ \frac{g^2 \pi_i}{1-g^2\pi^2} + \delta_{ij} \right] (\partial_j \pi_j)} \langle (\cdots) \rangle$$

(5.2)

Observe that the Jacobian

$$\prod_{x \in \mathbb{R}^2} \det \sqrt{g_{ij}(x)} := \prod_{x \in \mathbb{R}^2} \frac{1}{\sqrt{1 - g^2\pi^2}}$$

$$= \exp \left[ -\frac{1}{2} \int_{\mathbb{R}^2} \frac{d^2x}{a^2} \ln \left( 1 - g^2\pi^2 \right) \right]$$

$$= \exp \left[ -\frac{1}{2} \int_{\mathbb{R}^2} d^2x \ln \left( 1 - g^2\pi^2 \right) \right]$$

$$= \exp \left[ -\delta(r = 0) \frac{1}{2} \int_{\mathbb{R}^2} d^2x \ln \left( 1 - g^2\pi^2 \right) \right]$$

(5.3)

depends explicitly on the short distance cut-off $a$ that regularizes the delta function in real space,

$$\delta(r) = \begin{cases} 
0, & \text{if } r \neq 0. \\
\frac{1}{a^2}, & \text{if } r = 0. 
\end{cases}$$

(5.4)

In the sequel I can forget the Heavyside function in the measure for the spin-waves as it plays no role in perturbation theory in powers of $g^2$.

To organize the perturbative expansion, note that I need to expand
Lecture notes from Dr. Christopher Mudry, class taught at ETHZ during spring 2011

- the argument of the expectation value in powers of \( g^2 \), i.e., I need

\[
\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{24}x^2 + \cdots . \tag{5.5}
\]

- the action in powers of \( g^2 \), i.e., I need

\[
\ln(1-x) = -x - \frac{1}{2}x^2 + \cdots . \tag{5.6}
\]

- the Boltzman weight in powers of \( g^2 \), i.e., I need

\[
e^{-x} = 1 - x + \frac{1}{2}x^2 + \cdots . \tag{5.7}
\]

- the inverse of the partition function in powers of \( g^2 \), i.e., I need

\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots . \tag{5.8}
\]

Expansion in powers of \( g^2 \) of the argument in the expectation value (5.1) gives

\[
\mathcal{C}_{sw,g^2,H=0}^{(1,1)}(x,y;a) = 1 + g^2 \left\langle -\frac{1}{2} \pi(x) \cdot \pi(x) - \frac{1}{2} \pi(y) \cdot \pi(y) + \pi(x) \cdot \pi(y) \right\rangle_{sw;a}
+ g^4 \left\langle \frac{1}{4} \pi^2(x) \pi^2(y) - \frac{1}{8} \pi^2(x) \pi^2(x) - \frac{1}{8} \pi^2(y) \pi^2(y) \right\rangle_{sw;a}
+ \mathcal{O}(g^6). \tag{5.9}
\]

Before proceeding with the expansion, I introduce the notation

\[
\mathcal{L}_{sw,a} := \delta(r=0) \frac{1}{2} \ln (1 - g^2 \pi^2) + \frac{1}{2} (\partial_\mu \pi_i) \left( \frac{g^2 \pi_i \pi_j}{1 - g^2 \pi^2} + \delta_{ij} \right) (\partial_\mu \pi_j)
\equiv \mathcal{L}_0 + g^2 \mathcal{L}_{1,1} + g^2 \mathcal{L}_{1,2;a} + \mathcal{O}(g^4), \tag{5.10a}
\]

where

\[
\mathcal{L}_0 := \frac{1}{2} (\partial_\mu \pi) \cdot (\partial_\mu \pi), \tag{5.10b}
\]

\[
\mathcal{L}_{1,1} := \frac{1}{2} (\pi \cdot \partial_\mu \pi) (\pi \cdot \partial_\mu \pi), \tag{5.10c}
\]

\[
\mathcal{L}_{1,2;a} := - \delta(r=0) \frac{1}{2} \pi^2, \tag{5.10d}
\]

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The four actions obtained from the four Lagrangians $L_{sw, a}$, $L_0$, $L_{1,1}$, and $L_{1,2,a}$, are denoted $S_{sw, a}$, $S_0$, $S_{1,1}$, and $S_{1,2,a}$, respectively. I will also need the expansion

$$g^2 \langle AB \rangle := g^2 \frac{\int D[\theta] e^{-S_0 - g^2 S_1} AB}{\int D[\theta] e^{-S_0 - g^2 S_1}}$$

$$= g^2 \frac{\int D[\theta] e^{-S_0} AB (1 - g^2 S_1 + \cdots)}{\int D[\theta] e^{-S_0} (1 - g^2 S_1 + \cdots)}$$

$$= g^2 \langle AB \rangle_0 + g^4 [\langle AB \rangle_0 + \langle AB \rangle_0 \langle S_1 \rangle_0] + O(g^6), \quad (5.11)$$

where $S_1 = S_{1,1} + S_{1,2,a}$ and

$$((\cdots)_0 := \frac{\int D[\theta] e^{-S_0} (\cdots)}{\int D[\theta] e^{-S_0}}. \quad (5.12)$$

Alltogether, the final expansion for the spin-spin correlator reads

$$G_{sw, g^2, H=0}^{(1,1)}(x, y; a) = 1$$

$$+ g^2 \left(-\frac{1}{2} \pi(x) \cdot \pi(x) - \frac{1}{2} \pi(y) \cdot \pi(y) + \pi(x) \cdot \pi(y)\right)_{S_0}$$

$$+ g^4 \left(\frac{1}{4} \pi^2(x) \pi^2(y) - \frac{1}{8} \pi^2(x) \pi^2(x) - \frac{1}{8} \pi^2(y) \pi^2(y)\right)_{S_0}$$

$$+ g^4 \left[\left(\frac{1}{2} \pi^2(x) + \frac{1}{2} \pi^2(y) - \pi(x) \cdot \pi(y)\right) (S_{1,1} + S_{1,2,a})\right]_{S_0}$$

$$+ g^4 \left[\left(-\frac{1}{2} \pi^2(x) - \frac{1}{2} \pi^2(y) + \pi(x) \cdot \pi(y)\right)\right]_{S_0} \langle(S_{1,1} + S_{1,2,a})\rangle_{S_0}$$

$$+ O(g^6). \quad (5.13)$$

This is the expansion of the two-point function in the $O(2)$ NLSM up to order $g^4$ in the coupling constant.

Recalling that the limit $g^2 \rightarrow 0$ corresponds to zero temperature according to Eq. (2.2), we see that the two-point function is constant to zero-th order in $g^2$. This is the signature of spontaneous-symmetry breaking through ferromagnetic LRO.

Spin-waves disturb the ferromagnetic LRO at any finite temperature. Noting that

$$\langle \pi_i(x) \pi_j(y) \rangle_{S_0} \rightarrow \delta_{ij} \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik(x-y)}}{k^2 + M^2}$$

$$= -\delta_{ij} \frac{1}{2\pi} \ln (M|x - y|) + \cdots$$

$$\equiv \delta_{ij} G(x, y) + \cdots, \quad i, j = 1, \cdots, N - 1, \quad (5.14)$$
we see that the deviations from ferromagnetic LRO induced by spin-waves are logarithmically large to order $g^2$,

\[
G^{(1,1)}_{sw g^2, H=0}(x, y; a) = 1 + g^2 \left( -\frac{1}{2} \pi(x) \cdot \pi(x) - \frac{1}{2} \pi(y) \cdot \pi(y) + \pi(x) \cdot \pi(y) \right)_{S_0} + \mathcal{O}(g^4)
\]

\[
= 1 + g^2 \left[ -\frac{1}{2} (N-1)G(x, x) - \frac{1}{2} (N-1)G(y, y) + (N-1)G(x, y) \right] + \mathcal{O}(g^4)
\]

\[
= 1 + g^2 (N-1) \left[ G(x, y) - G(0, 0) \right] + \mathcal{O}(g^4),
\]  

(5.15)

where it is understood that the IR cut-off $M$ drops out from

\[
G(x, y) - G(0, 0) = -\frac{1}{2\pi} \left[ \ln(M|x - y|) - \ln(Ma) \right] + \cdots
\]

\[
= -\frac{1}{2\pi} \ln \frac{|x - y|}{a} + \cdots.
\]  

(5.16)

Perturbation theory in powers of $g^2$ thus appear to be hopeless except for the possibility that the contribution of order $g^4$ to the expansion be proportional to

\[
[G(x, y) - G(0, 0)]^2.
\]  

(5.17)

Indeed, this possibility could signal that the inclusion of spin-waves renders the anomalous scaling dimensions of $\pi$ nonvanishing, as was the case for the $O(2)$ NLSM,\(^{13}\) and that an expansion in powers of $g^2$ could be reinterpreted in a sensible way through a RG analysis based on a Callan-Symanzik equation.

There are several contributions to account for to order $g^4$. The third line of Eq. (5.13) gives, with the application

\[
\langle ABCD \rangle_0 = \langle AB \rangle_0 \langle CD \rangle_0 + \langle AC \rangle_0 \langle BD \rangle_0 + \langle AD \rangle_0 \langle BC \rangle_0
\]  

(5.18)

\(^{13}\) This can be seen by expanding the right hand side of Eq. (3.59) in powers of $g^2$. 

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of Wick’s theorem and with the help of translation invariance,

\[
+ g^4 \left\langle + \frac{1}{4} \pi^2(x) \pi^2(y) - \frac{1}{8} \pi^2(x) \pi^2(x) - \frac{1}{8} \pi^2(y) \pi^2(y) \right\rangle_{S_0} = \\
+ g^4 \left[ + \frac{1}{4} \left\langle \pi^2(x) \right\rangle_{S_0} \left\langle \pi^2(y) \right\rangle_{S_0} + \frac{1}{2} \left\langle \pi_i(x) \pi_j(y) \right\rangle_{S_0} \left\langle \pi_i(x) \pi_j(y) \right\rangle_{S_0} \right] \\
+ g^4 \left[ - \frac{1}{8} \left\langle \pi^2(x) \right\rangle_{S_0} \left\langle \pi^2(x) \right\rangle_{S_0} - \frac{1}{4} \left\langle \pi_i(x) \pi_j(x) \right\rangle_{S_0} \left\langle \pi_i(x) \pi_j(x) \right\rangle_{S_0} \right] \\
+ g^4 \left[ - \frac{1}{8} \left\langle \pi^2(y) \right\rangle_{S_0} \left\langle \pi^2(y) \right\rangle_{S_0} - \frac{1}{4} \left\langle \pi_i(y) \pi_j(y) \right\rangle_{S_0} \left\langle \pi_i(y) \pi_j(y) \right\rangle_{S_0} \right] = \\
+ g^4 \left[ + \frac{1}{2} \left\langle \pi_i(x) \pi_j(y) \right\rangle_{S_0} \left\langle \pi_i(x) \pi_j(y) \right\rangle_{S_0} - \frac{1}{2} \left\langle \pi_i(0) \pi_j(0) \right\rangle_{S_0} \left\langle \pi_i(0) \pi_j(0) \right\rangle_{S_0} \right] = \\
+ g^4 \frac{1}{2} (N - 1) \left[ G^2(x, y) - G^2(0, 0) \right].
\]

The fourth line of Eq. (5.13) demands the evaluation of

\[
+ \frac{g^4}{4} \int \frac{d^2 r}{R^2} \left\langle \pi_i(x) \pi_i(x) \pi_j(r) \partial_\mu \pi_j(r) \right\rangle_{S_0} \\
+ \frac{g^4}{4} \int \frac{d^2 r}{R^2} \left\langle \pi_i(y) \pi_i(y) \pi_j(r) \partial_\mu \pi_j(r) \right\rangle_{S_0} \\
- \frac{g^4}{2} \int \frac{d^2 r}{R^2} \left\langle \pi_i(x) \pi_i(y) \pi_j(r) \partial_\mu \pi_j(r) \right\rangle_{S_0} \\
+ \frac{g^4}{4} \left( - \frac{1}{a^2} \right) \int \frac{d^2 r}{R^2} \left\langle \pi_i(x) \pi_i(x) \pi_j(r) \right\rangle_{S_0} \\
+ \frac{g^4}{4} \left( - \frac{1}{a^2} \right) \int \frac{d^2 r}{R^2} \left\langle \pi_i(y) \pi_i(y) \pi_j(r) \right\rangle_{S_0} \\
- \frac{g^4}{2} \left( - \frac{1}{a^2} \right) \int \frac{d^2 r}{R^2} \left\langle \pi_i(x) \pi_i(y) \pi_j(r) \right\rangle_{S_0}.
\]

(5.20)
The fifth line of Eq. (5.13) subtracts

\[ + \frac{g^4}{4} \int_{\mathbb{R}^2} d^2r \left\langle \pi_i (x) \pi_i (x) \right\rangle_{S_0} \left\langle \left[ \pi_j (r) \partial_\mu \pi_j (r) \right] \left[ \pi_k (r) \partial_\mu \pi_k (r) \right] \right\rangle_{S_0} \]

\[ + \frac{g^4}{4} \int_{\mathbb{R}^2} d^2r \left\langle \pi_i (y) \pi_i (y) \right\rangle_{S_0} \left\langle \left[ \pi_j (r) \partial_\mu \pi_j (r) \right] \left[ \pi_k (r) \partial_\mu \pi_k (r) \right] \right\rangle_{S_0} \]

\[ - \frac{g^4}{2} \int_{\mathbb{R}^2} d^2r \left\langle \pi_i (x) \pi_i (y) \right\rangle_{S_0} \left\langle \left[ \pi_j (r) \partial_\mu \pi_j (r) \right] \left[ \pi_k (r) \partial_\mu \pi_k (r) \right] \right\rangle_{S_0} \]

\[ + \frac{g^4}{4} \left( -\frac{1}{a^2} \right) \int_{\mathbb{R}^2} d^2r \left\langle \pi_i (x) \pi_i (x) \right\rangle_{S_0} \left\langle \pi_j (r) \pi_j (r) \right\rangle_{S_0} \]

\[ + \frac{g^4}{4} \left( -\frac{1}{a^2} \right) \int_{\mathbb{R}^2} d^2r \left\langle \pi_i (y) \pi_i (y) \right\rangle_{S_0} \left\langle \pi_j (r) \pi_j (r) \right\rangle_{S_0} \]

\[ - \frac{g^4}{2} \left( -\frac{1}{a^2} \right) \int_{\mathbb{R}^2} d^2r \left\langle \pi_i (x) \pi_i (y) \right\rangle_{S_0} \left\langle \pi_j (r) \pi_j (r) \right\rangle_{S_0} \]

\[ (5.21) \]

from the fourth line of Eq. (5.13), i.e., it is sufficient to evaluate Eq. (5.20) with the help of Wick’s theorem with the additional rule that no Wick contraction between the two points \( x \) and \( x \) or \( y \) and \( y \) or \( x \) and \( y \) can occur.\(^\text{14}\)

\(^{14}\) Wick’s theorem reduces a Gaussian expectation value \( \left\langle \cdots \right\rangle_0 \) of \( 2m \) variables to the sum over all possible products of two-point functions, say for \( m = 3 \),

\[ \left\langle ABCDEF \right\rangle_0 = \left\langle AB \right\rangle_0 \left\langle CDEF \right\rangle_0 + \left\langle AC \right\rangle_0 \left\langle BDEF \right\rangle_0 + \left\langle AD \right\rangle_0 \left\langle BCEF \right\rangle_0 + \left\langle AE \right\rangle_0 \left\langle BCDF \right\rangle_0 \]

\[ + \left\langle AF \right\rangle_0 \left\langle BCDE \right\rangle_0 \]

\[ (5.22) \]

where

\[ \left\langle ABCD \right\rangle_0 = \left\langle AB \right\rangle_0 \left\langle CD \right\rangle_0 + \left\langle AC \right\rangle_0 \left\langle BD \right\rangle_0 + \left\langle AD \right\rangle_0 \left\langle BC \right\rangle_0. \]

\[ (5.23) \]

There are thus \( 5 \times 3 = 15 \) contributions when \( m = 3 \). Subtraction from \( \left\langle ABCDEF \right\rangle_0 \) of \( \left\langle AB \right\rangle_0 \left\langle CDEF \right\rangle_0 \) gives 12 contributions.
Because of translation invariance, Eqs. (5.20) and (5.21) simplify to

\[ + \frac{g^4}{2} \int_{\mathbb{R}^2} d^2 r \langle \pi_i(0) \pi_i(0) [\pi_j(\mathbf{r}) \partial_\mu \pi_j(\mathbf{r})][\pi_k(\mathbf{r}) \partial_\mu \pi_k(\mathbf{r})] \rangle_{S_0} \]

\[ - \frac{g^4}{2} \int_{\mathbb{R}^2} d^2 r \langle \pi_i(\mathbf{x}) \pi_i(\mathbf{y}) [\pi_j(\mathbf{r}) \partial_\mu \pi_j(\mathbf{r})][\pi_k(\mathbf{r}) \partial_\mu \pi_k(\mathbf{r})] \rangle_{S_0} \]

\[ + \frac{g^4}{2} \left( - \frac{1}{a^2} \right) \int_{\mathbb{R}^2} d^2 r \langle \pi_i(0) \pi_i(0) \pi_j(\mathbf{r}) \pi_j(\mathbf{r}) \rangle_{S_0} \]

\[ - \frac{g^4}{2} \left( - \frac{1}{a^2} \right) \int_{\mathbb{R}^2} d^2 r \langle \pi_i(\mathbf{x}) \pi_i(\mathbf{y}) \pi_j(\mathbf{r}) \pi_j(\mathbf{r}) \rangle_{S_0} \]

\[ (5.24) \]

and

\[ + \frac{g^4}{2} \int_{\mathbb{R}^2} d^2 r \langle \pi_i(0) \pi_i(0) \rangle_{S_0} \langle [\pi_j(\mathbf{r}) \partial_\mu \pi_j(\mathbf{r})][\pi_k(\mathbf{r}) \partial_\mu \pi_k(\mathbf{r})] \rangle_{S_0} \]

\[ - \frac{g^4}{2} \int_{\mathbb{R}^2} d^2 r \langle \pi_i(\mathbf{x}) \pi_i(\mathbf{y}) \rangle_{S_0} \langle [\pi_j(\mathbf{r}) \partial_\mu \pi_j(\mathbf{r})][\pi_k(\mathbf{r}) \partial_\mu \pi_k(\mathbf{r})] \rangle_{S_0} \]

\[ + \frac{g^4}{2} \left( - \frac{1}{a^2} \right) \int_{\mathbb{R}^2} d^2 r \langle \pi_i(0) \pi_i(0) \rangle_{S_0} \langle \pi_j(\mathbf{r}) \pi_j(\mathbf{r}) \rangle_{S_0} \]

\[ - \frac{g^4}{2} \left( - \frac{1}{a^2} \right) \int_{\mathbb{R}^2} d^2 r \langle \pi_i(\mathbf{x}) \pi_i(\mathbf{y}) \rangle_{S_0} \langle \pi_j(\mathbf{r}) \pi_j(\mathbf{r}) \rangle_{S_0} \]

\[ (5.25) \]

respectively. It is then sufficient to evaluate

\[ - \frac{g^4}{2} \int_{\mathbb{R}^2} d^2 r \left[ \langle \pi_i(\mathbf{x}) \pi_i(\mathbf{y}) [\pi_j(\mathbf{r}) \partial_\mu \pi_j(\mathbf{r})][\pi_k(\mathbf{r}) \partial_\mu \pi_k(\mathbf{r})] \rangle_{S_0} \right. \]

\[ \left. - \langle \pi_i(\mathbf{x}) \pi_i(\mathbf{y}) \rangle_{S_0} \langle [\pi_j(\mathbf{r}) \partial_\mu \pi_j(\mathbf{r})][\pi_k(\mathbf{r}) \partial_\mu \pi_k(\mathbf{r})] \rangle_{S_0} \right] \]

\[ (5.26) \]

and

\[ - \frac{g^4}{2} \left( - \frac{1}{a^2} \right) \int_{\mathbb{R}^2} d^2 r \left[ \langle \pi_i(\mathbf{x}) \pi_i(\mathbf{y}) \pi_j(\mathbf{r}) \pi_j(\mathbf{r}) \rangle_{S_0} - \langle \pi_i(\mathbf{x}) \pi_i(\mathbf{y}) \rangle_{S_0} \langle \pi_j(\mathbf{r}) \pi_j(\mathbf{r}) \rangle_{S_0} \right] \]

\[ (5.27) \]

since one can always choose \( \mathbf{x} = \mathbf{y} \). Remarkably, contribution (5.27) is contained in contribution (5.26) but with the opposite sign and thus cancels out of the spin-spin correlator. To see this, make use of translation invariance and of Eqs. (5.22) and (5.23) to write the Wick
decomposition

\[-\frac{g^4}{2} \int d^2 r \left[ \langle \pi_i(x) \pi_i(y) | \pi_j(r) \partial_\mu \pi_j(r) \rangle [\pi_k(r) \partial_\mu \pi_k(r)] \right]_{S_0} \]

\[\quad - \langle \pi_i(x) \pi_i(y) \rangle_{S_0} \left[ \langle \pi_j(r) \partial_\mu \pi_j(r) \rangle [\pi_k(r) \partial_\mu \pi_k(r)] \right]_{S_0} = \]

\[-1 \times 2 \times \frac{g^4}{2} \int d^2 r \left[ \langle \pi_i(x) \pi_j(r) \rangle_{S_0} \langle [\pi_k(r) \partial_\mu \pi_k(r)] \rangle_{S_0} \langle \partial_\mu \pi_j(r) \rangle_{S_0} \right] \]

\[-1 \times 2 \times \frac{g^4}{2} \int d^2 r \langle \pi_i(x) \pi_j(r) \rangle_{S_0} \langle [\pi_k(r) \partial_\mu \pi_k(r)] \rangle_{S_0} \langle \partial_\mu \pi_j(r) \rangle_{S_0} \]

\[-1 \times 2 \times \frac{g^4}{2} \int d^2 r \langle \pi_i(y) \pi_j(r) \rangle_{S_0} \langle [\pi_k(r) \partial_\mu \pi_k(r)] \rangle_{S_0} \langle \partial_\mu \pi_j(r) \rangle_{S_0} \]

\[-1 \times 2 \times \frac{g^4}{2} \int d^2 r \langle \pi_i(y) \pi_j(r) \rangle_{S_0} \langle [\pi_k(r) \partial_\mu \pi_k(r)] \rangle_{S_0} \langle \partial_\mu \pi_j(r) \rangle_{S_0} \]

\[-1 \times 2 \times \frac{g^4}{2} \int d^2 r \langle \pi_i(x) \partial_\mu \pi_j(r) \rangle_{S_0} \langle \pi_j(r) \pi_k(r) \rangle_{S_0} \langle \partial_\mu \pi_k(r) \rangle_{S_0} \]

\[-1 \times 2 \times \frac{g^4}{2} \int d^2 r \langle \pi_i(x) \partial_\mu \pi_j(r) \rangle_{S_0} \langle \pi_j(r) \pi_k(r) \rangle_{S_0} \langle \partial_\mu \pi_k(r) \rangle_{S_0} \]

\[-1 \times 2 \times \frac{g^4}{2} \int d^2 r \langle \pi_i(x) \partial_\mu \pi_j(r) \rangle_{S_0} \langle [\pi_k(r) \partial_\mu \pi_k(r)] \rangle_{S_0} \langle \partial_\mu \pi_j(r) \rangle_{S_0}. \quad (5.28)\]

Insertion of the unperturbed Green function (5.14) turns Eq. (5.28) into

\[-\frac{g^4}{2} \int d^2 r \left[ \langle \pi_i(x) \pi_i(y) | \pi_j(r) \partial_\mu \pi_j(r) \rangle [\pi_k(r) \partial_\mu \pi_k(r)] \right]_{S_0} \]

\[\quad - \langle \pi_i(x) \pi_i(y) \rangle_{S_0} \left[ \langle \pi_j(r) \partial_\mu \pi_j(r) \rangle [\pi_k(r) \partial_\mu \pi_k(r)] \right]_{S_0} = \]

\[-1 \times 2 \times \frac{g^4}{2} \int d^2 r \delta_{ij} G(x, r) \delta_{kk} \lim_{r \to \bar{r}} \left( \partial_{\bar{r}_\mu} G \right)(r, \bar{r}) \delta_{ji} \left( \partial_{r_\mu} G \right)(r, y) \]

\[-1 \times 2 \times \frac{g^4}{2} \int d^2 r \delta_{ij} G(x, r) \delta_{kk} \lim_{r \to \bar{r}} \left( \partial_{\bar{r}_\mu} G \right)(r, \bar{r}) \delta_{ji} \left( \partial_{r_\mu} G \right)(r, x) \]

\[-1 \times 2 \times \frac{g^4}{2} \int d^2 r \delta_{ij} G(y, r) \delta_{kk} \lim_{r \to \bar{r}} \left( \partial_{\bar{r}_\mu} G \right)(r, \bar{r}) \delta_{ji} \left( \partial_{r_\mu} G \right)(r, y) \]

\[-1 \times 2 \times \frac{g^4}{2} \int d^2 r \delta_{ij} \left( \partial_{r_\mu} G \right)(x, r) \delta_{kk} \left( \partial_{r_\mu} G \right)(0, 0) \delta_{ji} \left( \partial_{\bar{r}_\mu} G \right)(r, \bar{r}) \delta_{ki} G(r, y). \quad (5.29)\]
The first four lines on the right hand side of Eq. (5.29) vanish since
\[
\lim_{\tilde{r}\to r} \left( \partial_{\tilde{r}_\mu} G \right)(r, \tilde{r}) \sim \int_{\mathbb{R}^2} \frac{d^2 \mathbf{q}}{(2\pi)^2} \frac{q_\mu}{q^2} \]
\[
= 0. \tag{5.30}
\]
The fifth line gives
\[
-1 \times 2 \times \frac{g^4}{2} \int_{\mathbb{R}^2} d^2 \mathbf{r} \delta_{ij} \left( \partial_{r_\mu} G \right)(x, r) \delta_{jk} G(0, 0) \delta_{ki} \left( \partial_{\tilde{r}_\mu} G \right)(r, y) =
\]
\[
-1 \times 2 \times \frac{g^4}{2} (N - 1) G(0, 0) \int_{\mathbb{R}^2} d^2 \mathbf{r} \left( \partial_{r_\mu} G \right)(x, r) \left( \partial_{\tilde{r}_\mu} G \right)(r, y) =
\]
\[
-1 \times 2 \times \frac{g^4}{2} (N - 1) G(0, 0) \int_{\mathbb{R}^2} d^2 \mathbf{r} G(x, r) \left[ (-) \partial_{r_\mu} \partial_{\tilde{r}_\mu} G \right](r, y) =
\]
\[
-\Delta G(r, y) = \delta(r - y) \Leftrightarrow q^2 G_q = 1
\]
\[
-1 \times 2 \times \frac{g^4}{2} (N - 1) G(0, 0) \int_{\mathbb{R}^2} d^2 \mathbf{r} G(x, r) \delta(r - y) =
\]
\[
-g^4 (N - 1) G(0, 0) G(x, y). \tag{5.31}
\]
The last line gives
\[
-1 \times 2 \times \frac{g^4}{2} \int_{\mathbb{R}^2} d^2 \mathbf{r} \delta_{ij} G(x, r) \delta_{jk} \lim_{\tilde{r}\to r} \left( \partial_{r_\mu} \partial_{\tilde{r}_\mu} G \right)(r, \tilde{r}) \delta_{ki} G(r, y) =
\]
\[
-g^4 (N - 1) \int_{\mathbb{R}^2} d^2 \mathbf{r} G(x, r) \lim_{\tilde{r}\to r} \delta(r - \tilde{r}) G(r, y) =
\]
\[
+g^4 (N - 1) \left( \frac{1}{a^2} \right) \int_{\mathbb{R}^2} d^2 \mathbf{r} G(x, r) G(r, y). \tag{5.32}
\]
This is nothing but the same as contribution (5.27) up to an overall sign. As promised contribution (5.27) cancels out. Adding up all nonvanishing contributions of order $g^4$ to the spin-spin correlator gives
\[
+ g^4 \frac{1}{2} (N - 1) \left[ G(x, y) - G(0, 0) \right]^2 = +g^4 \frac{1}{2} (N - 1) \left[ G^2(x, y) - G^2(0, 0) \right]
\]
\[
- g^4 (N - 1) G(0, 0) G(x, y)
\]
\[
+ g^4 (N - 1) G(0, 0) G(0, 0). \tag{5.33}
\]
In summary, the expansion of the spin-spin correlator up to order $g^4$ is
\[
G^{(1,1)}_{swg^2,H=0}(x,y;a) = 1 \\
+ g^2 (N - 1) [G(x,y) - G(0,0)] \\
+ g^4 \left( N - 1 \right) [G(x,y) - G(0,0)]^2 \\
+ \mathcal{O}(g^6).
\] (5.34)

As a check, we recognize the first two terms in the expansion in powers of $g^2$ of
\[
\left( \frac{a}{|x-y|} \right)^{\frac{g^2}{\pi}} = \exp \left( + g^2 [G(x,y) - G(0,0)] \right)
\] (5.35)
if we set $N = 2$. The origin of the divergent logarithms occurring in the expansion in powers of $g^2$ is the existence in two dimensions of very strong fluctuations. Spin-waves destroy ferromagnetic LRO. In mathematical terms, the engineering dimension of the spin degrees of freedom differs from the scaling dimension. Correspondingly, LRO is downgraded to QLRO.
