Infrared subtraction and factorisation beyond NLO

Paolo Torrielli

Università di Torino and INFN

PSI, 29/10/2019

based on Magnea, Maina, Pelliccioli, Signorile-Signorile, PT, Uccirati, 1806.09570, 1809.05444

Outline

- Motivation
- Warmup: new subtraction at NLO
- New subtraction at NNLO
- Factorisation and subtraction beyond NLO
- Outlook

Motivation

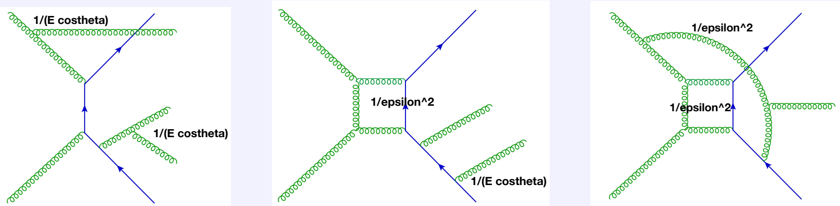
Need for accuracy at colliders

- The LHC is delivering highly accurate data, entering a high-precision phase.
- Theoretical precision is the best option to discover BSM at the LHC if it's there.
- An ambitious goal for the next years: automatic NNLO QCD.

Need for accuracy at colliders

- The LHC is delivering highly accurate data, entering a high-precision phase.
- Theoretical precision is the best option to discover BSM at the LHC if it's there.
- An ambitious goal for the next years: automatic NNLO QCD.
- Evaluation of two-loop amplitudes.
  - Progresses in massive $2 \to 2$ processes (see for example [Bonciani, et al.], [Melnikov, et al.], [Dunbar, et al.], ...), first steps in $2 \to 3$ massless [Badger, et al.], new ideas [Mastrolia, et al.].
- Cancellation of infrared infinities at NNLO.

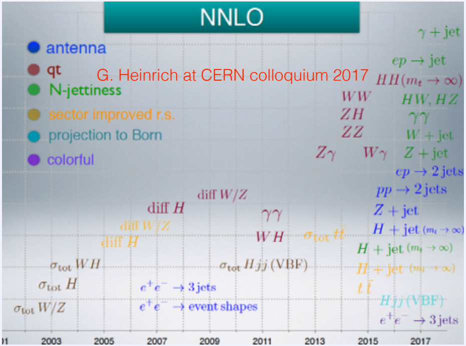
Infrared infinities at NNLO



$pp \to jjj$ at NNLO.

- Real and virtual amplitudes separately diverge in the IR limits.
- Only their sum, combined in IR-safe observables is finite by KLN theorem.
- Achieve efficient cancellation of infinities.

Cancellation of infinities beyond NLO



First results at N^{3}LO: $gg \to H$ [Anastasiou, et al.], DIS [Gehrmann, et al.], VBF(H) and VBF(HH) [Dreyer, Karlberg], $b\bar{b} \to H$ [Duhr, et al.].

- Many schemes on the market.
- Slicing: simpler but approximate. qT [Catani, Grazzini, Cieri, et al.], N-jettiness [Boughezal, Petriello, et al.], [Gaunt, Tackmann, et al.].
- Subtraction: more complex but exact. Antennae [Gehrmann, Glover, et al.], Stripper [Czakon, Mitov, et al.], nested soft-collinear [Caola, Melnikov, et al.], colourful [Del Duca, Troscanyi, et al.], projection to Born [Salam, et al.], sector decomposition [Anastasiou, et al.], [Binoth, et al.], $\mathcal{E}$-prescription [Frixione, Grazzini], FKS^2 in massive QED [Signer, et al.]
- New ideas: loop-tree duality [Rodrigo, Sborlini, et al.], FDR [Pittau], geometric subtraction [Herzog], loop approach [Anastasiou, Sterman].

Why to look for a new subtraction scheme at NNLO

- NNLO subtraction in QCD not yet solved in full generality.
  General? Automatable? Efficient? Local? Scaling with number of legs? ...
- Problem often tackled introducing radically new elements w.r.t. NLO solutions.
- Is there anything simpler? Are we using all freedom we have in defining subtraction?
- Can we hope to manage extensions (masses, higher orders) analytically?
- In the following, results on massless and final-state-only QCD partons.

Warmup: new subtraction at NLO

Subtracted NLO cross sections

- X = IRC safe, X_i = observable with i-body kinematics, $\delta_i \equiv \delta(X - X_i)$

$$\frac{d\sigma_{\mathrm{NLO}}}{dX} = \int d\Phi_n \, V \, \delta_n + \int d\Phi_{n+1} \, R \, \delta_{n+1} .$$

- Addends separately diverge: add and subtract local counterterm $\overline{K}$

$$\int d\Phi_{n+1} \, \overline{K} \, \delta_n \, .$$

- $\overline{K}$ = same singularities as R, locally in phase space, but simple enough to be integrated analytically in $d \neq 4$.

Subtracted NLO cross sections

- X = IRC safe, X_i = observable with i-body kinematics, $\delta_i \equiv \delta(X - X_i)$

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \, V \, \delta_n + \int d\Phi_{n+1} \, R \, \delta_{n+1} .$$

- Addends separately diverge: add and subtract local counterterm $\overline{K}$

$$\int d\Phi_{n+1} \, \overline{K} \, \delta_n \, .$$

- $\overline{K}$ = same singularities as R, locally in phase space, but simple enough to be integrated analytically in $d \neq 4$.

- Integrated counterterm in d dimensions:

$$I = \int d\Phi_{\rm rad} \, \overline{K} \, , \qquad d\Phi_{\rm rad} = d\Phi_{n+1} \, / \, d\Phi_n .$$

- Subtracted $\mathcal{O}(\alpha_S)$ cross section

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \, (V + I) \, \delta_n + \int d\Phi_{n+1} (R \, \delta_{n+1} - \overline{K} \, \delta_n) .$$

- Integrals $\int (V + I)$ and $\int (R - \overline{K})$ separately finite and evaluated numerically in $d = 4$.

NLO phase-space partitions

- Simplify the subtraction problem: treat as few singularities at a time as possible.

- Partition phase space Φ_{n+1} with sector functions $\mathcal{W}_{ij}$ [Frixione, Kunszt, Signer, 9512328]

  - normalised as $\sum_{i,\, j\neq i} \mathcal{W}_{ij} = 1$

  - $R\,\mathcal{W}_{ij}$ is singular only in one soft ($\mathbf{S}_i$) and one collinear ($\mathbf{C}_{ij}$) configuration

- Minimal singularity structure: only two partons can go soft/collinear in a given partition.

- Sum rules:

$$\mathbf{S}_i \sum_{k\neq i} \mathcal{W}_{ik} = 1\,, \qquad\qquad \mathbf{C}_{ij} \sum_{ab\,\in\,\mathrm{perm}(ij)} \mathcal{W}_{ab} = 1\,,$$

- Summing over all sectors sharing a given singularity, and taking that singular limit on the sum, the $\mathcal{W}$'s disappear. Key for simplifying analytic integration of $\overline{K}$.

Structure of NLO singularities

- Singularities in a sector known in terms of invariants $s_{ab} = 2\,k_a \cdot k_b$, without parametrising.
- $\mathbf{S}_i\,R$ ($\mathbf{C}_{ij}\,R$) = leading term in R as $k_i^\mu \to 0$ (relative $k_\perp^\mu \to 0$).

$$\mathbf{S}_i\,R(\{k\}) = -\mathcal{N}_1 \sum_{l,\,m} \delta_{f_i g} \frac{s_{lm}}{s_{il}\,s_{im}} B_{lm}(\{k\}_{\not i}) , \qquad B_{lm} = \text{colour-correlated Born}$$

$$\mathbf{C}_{ij}\,R(\{k\}) = \frac{\mathcal{N}_1}{s_{ij}} P_{ij}^{\mu\nu}(s_{ir}, s_{jr})\, B_{\mu\nu}\Big(\{k\}_{\not i \not j}, k\Big) , \qquad B_{\mu\nu} = \text{spin-correlated Born}$$

$$\mathbf{S}_i\,\mathbf{C}_{ij}\,R(\{k\}) = 2\,\mathcal{N}_1\, C_{f_j}\, \delta_{f_i g} \frac{s_{jr}}{s_{ij}\,s_{ir}} B(\{k\}_{\not i}) .$$

- Define a candidate counterterm in sector ij as soft + collinear – overlap:

$$K_{ij} = (\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i\,\mathbf{C}_{ij}) R\, \mathcal{W}_{ij}$$

(limits applied to both R and $\mathcal{W}_{ij}$), limits commute.

- As minimal as FKS, but not yet parametrised: freedom to be exploited for analytic integration.

Mapping to Born kinematics

- Momentum mapping $\{k_1, ..., k_{n+1}\} \to \{\bar{k}_1, ..., \bar{k}_n\}$ to factorise Born phase space.

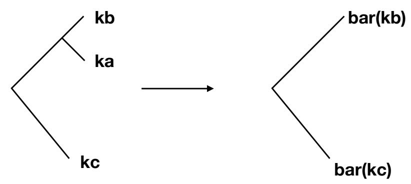
- Catani-Seymour [Catani, Seymour, 9605323] final-state dipole mapping $\{k\} \to \{\bar{k}\}^{(abc)}$:

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac} + s_{bc}} k_c \,,$$

$$\bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac} + s_{bc}} k_c \,,$$

$$s_{abc} = s_{ab} + s_{ac} + s_{bc},$$

$$\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c \,.$$



- Phase-space factorisation and parametrisation:

$$d\Phi_{n+1} = d\Phi_n^{(abc)} \times d\Phi_{\rm rad}^{(abc)} \equiv d\Phi_n^{(abc)} \times d\Phi_{\rm rad}\left(\bar{s}_{bc}^{(abc)}; y, z, \phi\right),$$

$$d\Phi_{\rm rad}^{(abc)} \propto (\bar{s}_{bc}^{(abc)})^{1-\epsilon} \int_0^\pi d\phi \, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz \Big[y(1-y)^2 \, z(1-z) \Big]^{-\epsilon} (1-y) \,,$$

$$s_{ab} = y \, s_{abc} \,, \qquad s_{ac} = z(1-y)\, s_{abc} \,, \qquad s_{bc} = (1-z)(1-y)\, s_{abc} \,.$$

Local-counterterm definition

- $\{k\} \to \{\bar{k}\}^{(abc)}$: adapt mapping to the invariants appearing in the kernels.
- $\mathbf{C}_{ij}\, R$ features invariants s_{ij}, s_{ir}, and s_{jr}: dipole = (ijr).

  Each term in the eikonal sum in $\mathbf{S}_i\, R$ features s_{il}, s_{im}, and s_{lm}: dipole = (ilm).
- Remapped singular limits:

$$
\begin{aligned}
\overline{\mathbf{S}}_i\, R(\{k\}) &= -\mathcal{N}_1 \sum_{l,\,m} \delta_{f_i g}\, \frac{s_{lm}}{s_{il}\, s_{im}}\, B_{lm}\Big(\{\bar{k}\}^{(ilm)}\Big)\,, \\
\overline{\mathbf{C}}_{ij}\, R(k) &= \frac{\mathcal{N}_1}{s_{ij}}\, P_{ij}^{\mu\nu}(s_{ir}, s_{jr})\, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)\,, \\
\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R(\{k\}) &= 2\,\mathcal{N}_1\, C_{f_j}\, \delta_{f_i g}\, \frac{s_{jr}}{s_{ij}\, s_{ir}}\, B\Big(\{\bar{k}\}^{(ijr)}\Big)\,,
\end{aligned}
$$

- Local-counterterm:

$$
\overline{K}_{ij} \equiv \Big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\Big)\, R\, \mathcal{W}_{ij}\,, \qquad \overline{K} = \sum_{i,j\neq i} \overline{K}_{ij}\,,
$$

NLO-counterterm integration

$$\overline{K} = \sum_{i,j\neq i} \overline{K}_{ij} = \sum_{i} \bar{\mathbf{S}}_i\, R + \sum_{i,\, j>i} \bar{\mathbf{C}}_{ij} \left(1 - \bar{\mathbf{S}}_i - \bar{\mathbf{S}}_j\right) R\,.$$

- Benefit from sum rules: eliminate $\mathcal{W}_{ij}$ from $\overline{K}$ (as in FKS).
- Benefit from mapping adaption: each integrand has a trivial phase space (as in CS).
- Soft integration (y and z = CS variables for dipole (ilm)):

$$I^{\,\mathrm{s}} = -\mathcal{N}_1 \frac{\varsigma_{n+1}}{\varsigma_n} \sum_i \delta_{f_i g} \sum_{\substack{l\neq i\\ m\neq i}} B_{lm}\left(\{\bar{k}\}^{(ilm)}\right) \frac{1}{\bar{s}_{lm}^{(ilm)}} \int d\Phi_{\mathrm{rad}} \left(\bar{s}_{lm}^{(ilm)}; y, z, \phi\right) \frac{1-z}{yz}$$

$$= -\mathcal{N}_1 \frac{\varsigma_{n+1}}{\varsigma_n} \sum_i \delta_{f_i g} \sum_{\substack{l\neq i\\ m\neq i}} B_{lm}\left(\{\bar{k}\}^{(ilm)}\right) \frac{(4\pi)^{\epsilon-2}}{\left(\bar{s}_{lm}^{(ilm)}\right)^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)}\,.$$

NLO-counterterm integration

- Full result, including hard-collinear

$$
\begin{aligned}
I(\{\bar{k}\}) \quad = \quad & -\mathcal{N}_1 \sum_{l,\, m\neq l} \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{lm}^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \, B_{lm}\big(\{\bar{k}\}\big) \\
& -\mathcal{N}_1 \sum_{p} \frac{(4\pi)^{\epsilon-2}}{\bar{s}_{pr}^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)} \, \mathbb{C}\, B\big(\{\bar{k}\}\big) \,,
\end{aligned}
$$

with $\mathbb{C} = \frac{C_A+4\,T_R\,N_f}{2(3-2\epsilon)}\,\delta_{f_p g} + \frac{C_F}{2}\,\delta_{f_p\{q,\bar{q}\}}$.

- Result exact in ϵ. Not important *per se*, but a sign of simplicity.

- Virtual ϵ poles analytically reproduced in general.

- Finite parts checked differentially in a variety of cases.

Lessons from NLO

- Partition functions and their sum rules are convenient tools, as in FKS.
- Term-by-term mapping adaption as in CS $\implies$ simplifications in analytic integration.
- Bridge between FKS and CS:
  - $\overline{K}_{ij}$ is like FKS, but with adapted parametrisation.
  - $\overline{K} = \sum_{ij} \overline{K}_{ij}$ is like CS, but much simpler.
- Features to be exported to NNLO.

New subtraction at NNLO

Subtracted NNLO cross sections

- X = IRC safe, X_i = observable with i-body kinematics, $\delta_i \equiv \delta(X - X_i)$

$$\frac{d\sigma_{\text{NNLO}}}{dX} = \int d\Phi_n \, VV \, \delta_n + \int d\Phi_{n+1} \, RV \, \delta_{n+1} + \int d\Phi_{n+2} \, RR \, \delta_{n+2} .$$

- Add and subtract local counterterms:

$$\int d\Phi_{n+2} \, \overline{K}^{\,(\mathbf{1})} \, \delta_{n+1} \, , \qquad \int d\Phi_{n+2} \, (\overline{K}^{\,(\mathbf{2})} + \overline{K}^{\,(\mathbf{12})}) \, \delta_n \, , \qquad \int d\Phi_{n+1} \, \overline{K}^{(\mathbf{RV})} \, \delta_n \, .$$

- $\overline{K}^{\,(\mathbf{1})}$ and $(\overline{K}^{\,(\mathbf{2})} + \overline{K}^{\,(\mathbf{12})})$: same single- and double-unresolved singularities as RR.
  $\overline{K}^{\,(\mathbf{2})} \;\rightarrow\;$ double-unresolved limits (i.e. democratic);
  $\overline{K}^{\,(\mathbf{12})} \;\rightarrow\;$ single-unresolved limits of double-unresolved ones (i.e. strongly ordered);
  $\overline{K}^{(\mathbf{RV})} \;\rightarrow\;$ same phase-space singularities as RV.

- d-dimensional integrated counterterms ($d\Phi_{\mathrm{rad},i} \; = \; d\Phi_{n+2} \, / \, d\Phi_{n+2-i}$):

$$I^{(\mathbf{i})} = \int d\Phi_{\mathrm{rad},i} \, \overline{K}^{(\mathbf{i})} \, , \qquad I^{\,(\mathbf{12})} = \int d\Phi_{\mathrm{rad},1} \, \overline{K}^{\,(\mathbf{12})} \, , \qquad I^{(\mathbf{RV})} = \int d\Phi_{\mathrm{rad}} \, \overline{K}^{(\mathbf{RV})} \, ,$$

Subtracted NNLO cross sections

- Subtracted $\mathcal{O}(\alpha_S^2)$ cross section:

$$
\begin{aligned}
\frac{d\sigma_{\text{NNLO}}}{dX} = & \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathbf{RV})} \right) \delta_n \\
& + \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{n+1} - \left(\overline{K}^{(\mathbf{RV})} - I^{(12)} \right) \delta_n \right] \\
& + \int d\Phi_{n+2} \left[RR\, \delta_{n+2} - \overline{K}^{(1)} \delta_{n+1} - \left(\overline{K}^{(2)} + \overline{K}^{(12)} \right) \delta_n \right] .
\end{aligned}
$$

- Each line separately finite and evaluated numerically in $d = 4$.

- Singularity-cancellation pattern:
  - $RR - \overline{K}^{(1)} - (\overline{K}^{(2)} + \overline{K}^{(12)})$ finite in $d = 4$, and in Φ_{n+2}.
  - $RV + I^{(1)}$ finite in $d = 4$, but singular in Φ_{n+1}.
  - $\overline{K}^{(\mathbf{RV})} - I^{(12)}$ finite in $d = 4$, but singular in Φ_{n+1}.
  - $(RV + I^{(1)}) - (\overline{K}^{(\mathbf{RV})} - I^{(12)})$ finite in $d = 4$, and in Φ_{n+1}.
  - $VV + I^{(2)} + I^{(\mathbf{RV})}$ finite in $d = 4$, and in Φ_n.

NNLO phase-space partitions

- Partition Φ_{n+2} with sector functions $\mathcal{W}_{ijkl}$, (normalised as $\sum_{ijkl} \mathcal{W}_{ijkl} = 1$), to select as few singularities at a time as possible.

- Sum rules in double-unresolved limits: by summing over all sectors sharing the same singularity, and taking that singular limit on the sum, $\mathcal{W}$ functions must disappear. Key for analytic integration of double-unresolved counterterms.

- Factorisation properties: in the single-unresolved limits, NNLO $\mathcal{W}$'s factorise NLO ones.

$$\mathbf{C}_{ij}\,\mathcal{W}_{ijkl} \;\sim\; \mathcal{W}_{kl}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}\,, \qquad\qquad \mathbf{S}_{i}\,\mathcal{W}_{ijkl} \;\sim\; \mathcal{W}_{kl}\,\mathbf{S}_{i}\,\mathcal{W}_{ij}\,.$$

  Key for explicit cancellation of real-virtual poles in each $\mathcal{W}_{ij}$.

NNLO counterterms

- In each sector candidate counterterms collect singular limits of $RR\,\mathcal{W}$, written in terms of s_{ab}.

- Example for sector $\mathcal{W}_{ijkj}$ (where nonzero limits are $\mathbf{S}_i$, $\mathbf{C}_{ij}$, $\mathbf{S}_{ik}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$, $\mathbf{CS}_{ijk}$):

$$
\begin{aligned}
K^{(\mathbf{1})}_{ijkj} &= \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\, RR\,\mathcal{W}_{ijkj}\,, \\
K^{(\mathbf{2})}_{ijkj} &= \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) \\
&\qquad + \mathbf{CS}_{ijk}(1-\mathbf{SC}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big]\, RR\,\mathcal{W}_{ijkj}\,, \\
K^{(\mathbf{12})}_{ijkj} &= -\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) \\
&\qquad + \mathbf{CS}_{ijk}(1-\mathbf{SC}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big]\, RR\,\mathcal{W}_{ijkj}\,.
\end{aligned}
$$

- $\mathbf{S}_{ij}\,RR$, $\mathbf{C}_{ikj}\,RR$, and $\mathbf{SC}_{ijk}\,RR$ are universal kernels [Catani, Grazzini, 9810389, 9908523], [Campbell, Glover, 9710255], [Berends, Giele, 1989]. All limits commute.

NNLO-counterterm simplifications

- Simplifications possible, thanks to idempotency relations

$$(1-\mathbf{S}_i)\,\mathbf{SC}_{icd}\,RR\,\mathcal{W}_{ibcd}=0\,,\qquad (1-\mathbf{C}_{ij})\,\mathbf{CS}_{ijk}\,RR\,\mathcal{W}_{ijkd}=0\,.$$

$$\begin{aligned} K^{(\mathbf{2})}_{ijkj} &= \Big[\mathbf{S}_{ik}+\mathbf{C}_{ijk}(1-\mathbf{S}_{ik})+\mathbf{SC}_{ijk}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) \\ &\qquad +\mathbf{CS}_{ijk}(1-\mathbf{SC}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big]\,RR\,\mathcal{W}_{ijkj}\,, \\ K^{(\mathbf{12})}_{ijkj} &= -\Big[\mathbf{S}_{i}+\mathbf{C}_{ij}(1-\mathbf{S}_{i})\Big]\Big[\mathbf{S}_{ik}+\mathbf{C}_{ijk}(1-\mathbf{S}_{ik})+\mathbf{SC}_{ijk}(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) \\ &\qquad +\mathbf{CS}_{ijk}(1-\mathbf{SC}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big]\,RR\,\mathcal{W}_{ijkj}\,. \end{aligned}$$

- Limits **SC** and **CS** disappear from $K^{(\mathbf{2})}+K^{(\mathbf{12})}$ (see also [Caola, Melnikov, Roentsch]):

$$K^{(\mathbf{2})}_{ijkj}+K^{(\mathbf{12})}_{ijkj} = (1-\mathbf{S}_i)(1-\mathbf{C}_{ij})\Big[\mathbf{S}_{ik}+\mathbf{C}_{ijk}(1-\mathbf{S}_{ik})\Big]\,RR\,\mathcal{W}_{ijkj}\,,$$

very minimal structure!

- Complexity parallelised: more complicated processes need more partitions, but the intrinsic complexity in each partition does not scale.

Counterterms $\overline{K}^{(\mathbf{1})}$ and $\overline{K}^{(\mathbf{12})}$

- Use factorisation properties of $\mathcal{W}_{abcd}$, and sum rules of $\mathcal{W}_{ab}$:

$$\overline{K}^{(\mathbf{1})} = \sum_{k,l} \overline{\mathcal{W}}_{kl} \left[\sum_{i,\, j>i} \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\right) RR + \sum_i \overline{\mathbf{S}}_i \, RR \right] = \sum_{k,l} \overline{K}^{(\mathbf{1})}_{kl} \, .$$

in each NLO sector

full structure of single-unres. singularities

- Same integral as at NLO (known to all orders in ϵ).

- $RV\, \overline{\mathcal{W}}_{kl} + I^{(\mathbf{1})}_{kl}$ finite in $d = 4$ sector by sector in the NLO phase space.

- One can show that $I^{(\mathbf{12})}_{kl} = \left[\overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{kl} \left(1 - \overline{\mathbf{S}}_k\right) \right] I^{(\mathbf{1})}_{kl}$.

- $\overline{K}^{(\mathbf{RV})}_{kl} - I^{(\mathbf{12})}_{kl}$ finite in $d = 4$ sector by sector in the NLO phase space.

Counterterm $\overline{K}^{(\mathbf{2})}$

- Using sum rules, $\mathcal{W}$'s disappear from $\overline{K}^{(\mathbf{2})}$ and from its integral $I^{(\mathbf{2})}$. In the end:

$$\overline{K}^{(\mathbf{2})} = \sum_{i} \Bigg\{ \sum_{j>i} \overline{\mathbf{S}}_{ij} + \sum_{j>i} \sum_{k>j} \overline{\mathbf{C}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right)$$
$$+ \sum_{j>i} \sum_{\substack{k>i \\ k\neq j}} \sum_{\substack{l>k \\ l\neq j}} \overline{\mathbf{C}}_{ijkl} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl}\right)$$
$$+ \sum_{j\neq i} \sum_{\substack{k\neq i \\ k>j}} \overline{\mathbf{SC}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik}\right) \left(1 - \overline{\mathbf{C}}_{ijk} - \sum_{l\neq i,j,k} \overline{\mathbf{C}}_{iljk}\right)$$
$$+ \sum_{j>i} \sum_{k\neq i,j} \overline{\mathbf{CS}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) \left(1 - \overline{\mathbf{C}}_{ijk} - \sum_{l\neq i,j,k} \overline{\mathbf{C}}_{ijkl}\right) \Bigg\} RR \,,$$

- Analytic integration of a set of universal NNLO kernels with no $\mathcal{W}$ functions.

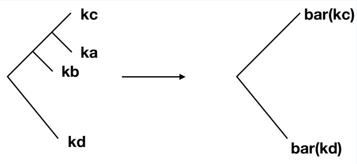
- As at NLO, mapping adaption to ease analytic integration.

Mappings from NNLO to Born kinematics

- $\{k\} \to \{\bar{k}\}^{(abcd)}$ mapping example, $d\Phi_{n+2} \;=\; d\Phi_n^{(abcd)} \times d\Phi_{\mathrm{rad},2}^{(abcd)}$

$$\bar{k}_c^{(abcd)} = k_a + k_b + k_c - \frac{s_{abc}}{s_{ad}+s_{bd}+s_{cd}}\, k_d\,,$$

$$\bar{k}_d^{(abcd)} = \frac{s_{abcd}}{s_{ad}+s_{bd}+s_{cd}}\, k_d\,,$$



$$s_{abcd} = s_{ab}+s_{ac}+s_{ad}+s_{bc}+s_{bd}+s_{cd}$$

$$\bar{k}_c^{(abcd)} + \bar{k}_d^{(abcd)} = k_a + k_b + k_c + k_d.$$

- This is used in double-collinear $\overline{\mathbf{C}}_{ijk}\, RR$ and double-soft $\overline{\mathbf{S}}_{ij}\, RR$ counterterms:

$$\overline{\mathbf{S}}_{ij}\, RR \;=\; \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \mathcal{I}_{cd}^{(ij)} B_{cd}\left(\{\bar{k}\}^{(ijcd)}\right) + \ldots\,,$$

$$\overline{\mathbf{C}}_{ijk}\, RR \;=\; \frac{\mathcal{N}_1^2}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}\, B_{\mu\nu}\left(\{\bar{k}\}^{(ijkr)}\right).$$

Analytic integration of $\overline{K}^{(\mathbf{2})}$

- All counterterms have been integrated analytically with no IBPs: mapping adaption works!

$$I^{(\mathbf{2})} = I^{(\mathbf{2})}_{\rm ss} + I^{(\mathbf{2})}_{\rm hcc} + \boxed{I^{(\mathbf{2})}_{\rm cc4} + I^{(\mathbf{2})}_{\rm sc3}}$$

Factorised: complexity = NLO x NLO

Analytic integration of $\overline{K}^{(\mathbf{2})}$

- All counterterms have been integrated analytically with no IBPs: mapping adaption works!

$$I^{(\mathbf{2})} = I^{(\mathbf{2})}_{\rm ss} + I^{(\mathbf{2})}_{\rm hcc} + I^{(\mathbf{2})}_{\rm cc4} + I^{(\mathbf{2})}_{\rm sc3}$$

$$\begin{aligned} I^{(\mathbf{2})}_{\rm ss} = \left(\frac{\alpha_{\rm S}}{4\pi}\right)^2 \left(\frac{\mu^2}{s}\right)^{2\epsilon} \Bigg\{ & \left[2\Big(\sum_{a,b} C_{f_a} C_{f_b}\Big) I^{\rm ss}_{C_f C_f} + 8\Big(\sum_a C^2_{f_a}\Big) I^{\rm ss}_{C_f^2} \right. \\ & \left. - \Big(\sum_a C_{f_a}\Big)\Big(N_f\, T_R\, I^{\rm ss}_{C_f T_R} - \frac{C_A}{2} I^{\rm ss}_{C_f C_A}\Big)\right] B(\{\bar{k}\}) \\ & + 2 \sum_{c,d\neq c} \left[-2\Big(\sum_a C_{f_a}\Big) I^{\rm ss}_{C_f B_{cd}} + N_f\, T_R\, I^{\rm ss}_{T_R B_{cd}} - \frac{C_A}{2} I^{\rm ss}_{C_A B_{cd}} \right] B_{cd}(\{\bar{k}\}) \\ & + 2 \sum_{c,d\neq c} I^{\rm ss}_{B_{cdcd}} B_{cdcd}(\{\bar{k}\}) + 4 \sum_{c,d\neq c} \sum_{e\neq d} I^{\rm ss}_{B_{cded}} B_{cded}(\{\bar{k}\}) \\ & + \sum_{c,d\neq c} \sum_{e,f\neq e} I^{\rm ss}_{B_{cdef}} B_{cdef}(\{\bar{k}\}) + \mathcal{O}(\epsilon) \Bigg\} \end{aligned}$$

Analytic integration of $\overline{K}^{(\mathbf{2})}$

- All counterterms have been integrated analytically with no IBPs: mapping adaption works!

$I^{(\mathbf{2})}_{\rm ss}$

$$I^{\rm ss}_{C_f C_f} = \frac{1}{\epsilon^4} + \frac{4}{\epsilon^3} + \left(16 - \frac{7}{6}\pi^2\right)\frac{1}{\epsilon^2} + \left(60 - \frac{14}{3}\pi^2 - \frac{50}{3}\zeta(3)\right)\frac{1}{\epsilon} + 216 - \frac{56}{3}\pi^2 - \frac{200}{3}\zeta(3) + \frac{29}{120}\pi^4$$

$$I^{\rm ss}_{C_f^2} = \left(1 - \frac{\pi^2}{6}\right)\frac{1}{\epsilon^2} + \left(10 - \frac{2}{3}\pi^2 - 6\,\zeta(3)\right)\frac{1}{\epsilon} + 68 - 4\,\pi^2 - 24\,\zeta(3) - \frac{7}{72}\pi^4$$

$$I^{\rm ss}_{C_f T_R} = \frac{2}{3}\frac{1}{\epsilon^3} + \frac{34}{9}\frac{1}{\epsilon^2} + \left(\frac{464}{27} - \frac{7}{9}\pi^2\right)\frac{1}{\epsilon} + \frac{5896}{81} - \frac{131}{27}\pi^2 - \frac{76}{9}\zeta(3)$$

$$I^{\rm ss}_{C_f C_A} = \frac{2}{\epsilon^4} + \frac{35}{3}\frac{1}{\epsilon^3} + \left(\frac{487}{9} - \frac{8}{3}\pi^2\right)\frac{1}{\epsilon^2} + \left(\frac{6248}{27} - \frac{269}{18}\pi^2 - \frac{154}{3}\zeta(3)\right)\frac{1}{\epsilon} + \frac{77404}{81} - \frac{3829}{54}\pi^2 - \frac{2050}{9}\zeta(3) - \frac{23}{60}\pi^4$$

$$I^{\rm ss}_{C_f B_{cd}} = \ln\frac{\bar s_{cd}}{s}\Bigg[-\frac{1}{\epsilon^3} - \frac{4}{\epsilon^2} - \left(20 - \frac{11}{6}\pi^2\right)\frac{1}{\epsilon} - 100 + \frac{22}{3}\pi^2 + \frac{122}{3}\zeta(3) + \frac{1}{2}\ln\frac{\bar s_{cd}}{s}\left(\frac{1}{\epsilon^2} + \frac{4}{\epsilon} + 20 - \frac{11}{6}\pi^2\right) - \frac{1}{6}\ln^2\frac{\bar s_{cd}}{s}\left(\frac{1}{\epsilon} + 4\right) + \frac{1}{24}\ln^3\frac{\bar s_{cd}}{s}\Bigg]$$

$$I^{\rm ss}_{T_R B_{cd}} = \ln\frac{\bar s_{cd}}{s}\Bigg[-\frac{2}{3}\frac{1}{\epsilon^2} - \frac{34}{9}\frac{1}{\epsilon} - \frac{464}{27} + \frac{7}{9}\pi^2 + \ln\frac{\bar s_{cd}}{s}\left(\frac{2}{3}\frac{1}{\epsilon} + \frac{34}{9}\right) - \frac{4}{9}\ln^2\frac{\bar s_{cd}}{s}$$

$$I^{\rm ss}_{C_A B_{cd}} = \ln\frac{\bar s_{cd}}{s}\Bigg[-\frac{2}{\epsilon^3} - \frac{35}{3}\frac{1}{\epsilon^2} - \left(\frac{487}{9} - \frac{8}{3}\pi^2\right)\frac{1}{\epsilon} - \frac{6248}{27} + \frac{269}{18}\pi^2 + \frac{154}{3}\zeta(3) + \ln\frac{\bar s_{cd}}{s}\left(\frac{2}{\epsilon^2} + \frac{35}{3}\frac{1}{\epsilon} + \frac{487}{9} - \frac{8}{3}\pi^2\right) - \frac{2}{3}\ln^2\frac{\bar s_{cd}}{s}\left(\frac{2}{\epsilon} + \frac{35}{3}\right) + \frac{2}{3}\ln^3\frac{\bar s_{cd}}{s}\Bigg]$$

$$I^{\rm ss}_{B_{cdcd}} = -4\big(1 - \zeta(3)\big)\left(\frac{1}{\epsilon} - 2\ln\frac{\bar s_{cd}}{s}\right) - 40 - \frac{\pi^2}{3} + 12\,\zeta(3) + \frac{13}{36}\pi^4$$

$$I^{\rm ss}_{B_{cded}} = \ln\frac{\bar s_{cd}}{s}\ln\frac{\bar s_{ed}}{s}\left(1 - \frac{\pi^2}{6}\right),$$

$$I^{\rm ss}_{B_{cdef}} = \ln\frac{\bar s_{cd}}{s}\ln\frac{\bar s_{ef}}{s}\Bigg[\frac{1}{\epsilon^2} + \frac{4}{\epsilon} + 16 - \frac{7}{6}\pi^2 - \frac{1}{2}\left(\ln\frac{\bar s_{cd}}{s} + \ln\frac{\bar s_{ef}}{s}\right)\left(\frac{1}{\epsilon} + 4\right) + \frac{1}{6}\left(\ln^2\frac{\bar s_{cd}}{s} + \ln^2\frac{\bar s_{ef}}{s}\right) + \frac{1}{4}\ln\frac{\bar s_{cd}}{s}\ln\frac{\bar s_{ef}}{s}\Bigg]$$

$(\{\bar k\})$

Analytic integration of $\overline{K}^{(\mathbf{2})}$

- All counterterms have been integrated analytically with no IBPs: mapping adaption works!

$$I^{(\mathbf{2})} = I^{(\mathbf{2})}_{\rm ss} + I^{(\mathbf{2})}_{\rm hcc} + I^{(\mathbf{2})}_{\rm cc4} + I^{(\mathbf{2})}_{\rm sc3}$$

$$I^{(\mathbf{2})}_{\rm hcc} = 2\left(\frac{\alpha_{\rm s}}{4\pi}\right)^2\left(\frac{\mu^2}{s}\right)^{2\epsilon}\sum_p \Big\{ \delta_{f_p g}\Big[N_f\,C_F\,T_R\,I^{\rm hcc}_{C_F g} + N_f\,C_A\,T_R\,I^{\rm hcc}_{C_A} + C_A^2\,I^{\rm hcc}_{C_A^2}\Big]$$
$$+\,\delta_{f_p\{q,\bar q\}}\,C_F\Big[N_f\,T_R\,I^{\rm hcc}_{C_F q} + C_F\,I^{\rm hcc}_{C_F^2} + C_A\,I^{\rm hcc}_{C_F C_A}\Big] + \mathcal{O}(\epsilon)\Big\}\,B(\{\bar k\})$$

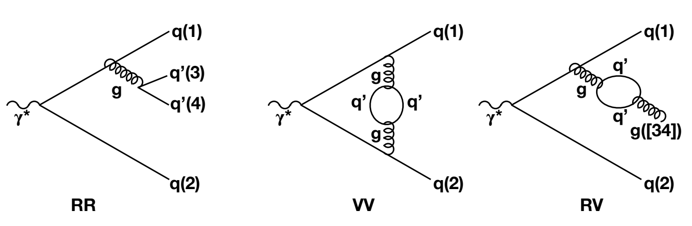
Analytic integration of $\overline{K}^{(2)}$

- All counterterms have been integrated analytically with no IBPs: mapping adaption works!

$$
\begin{aligned}
I^{\rm hcc}_{C_F g} &= -\frac{4}{3}\frac{1}{\epsilon^3} - \frac{62}{9}\frac{1}{\epsilon^2} + \left(-\frac{889}{27} + 2\,\pi^2\right)\frac{1}{\epsilon} - \frac{23833}{162} + \frac{31}{3}\,\pi^2 + \frac{320}{9}\,\zeta(3) \\
&\quad + 2\,\ln\frac{\bar{s}_{pr}}{s}\left(\frac{4}{3}\frac{1}{\epsilon^2} + \frac{62}{9}\frac{1}{\epsilon} + \frac{889}{27} - 2\,\pi^2\right) + 2\,\ln^2\frac{\bar{s}_{pr}}{s}\left(-\frac{4}{3}\frac{1}{\epsilon} - \frac{62}{9}\right) + \frac{16}{9}\,\ln^3\frac{\bar{s}_{pr}}{s} \\
I^{\rm hcc}_{C_A} &= -\frac{2}{\epsilon^3} - \frac{89}{9}\frac{1}{\epsilon^2} + \left(-\frac{1211}{27} + 3\,\pi^2\right)\frac{1}{\epsilon} - \frac{5240}{27} + \frac{89}{6}\,\pi^2 + \frac{160}{3}\,\zeta(3) \\
&\quad + 2\,\ln\frac{\bar{s}_{pr}}{s}\left(\frac{2}{\epsilon^2} + \frac{89}{9}\frac{1}{\epsilon} + \frac{1211}{27} - 3\,\pi^2\right) + 2\,\ln^2\frac{\bar{s}_{pr}}{s}\left(-\frac{2}{\epsilon} - \frac{89}{9}\right) - \frac{8}{3}\,\ln^3\frac{\bar{s}_{pr}}{s} \\
I^{\rm hcc}_{C_A^2} &= -\frac{5}{6}\frac{1}{\epsilon^3} - \frac{77}{18}\frac{1}{\epsilon^2} + \left(-16 + \frac{11}{12}\pi^2 - \zeta(3)\right)\frac{1}{\epsilon} - \frac{16943}{324} + \frac{61}{12}\,\pi^2 + \frac{56}{9}\zeta(3) - \frac{3}{40}\,\pi^4 \\
&\quad + 2\,\ln\frac{\bar{s}_{pr}}{s}\left(\frac{5}{6}\frac{1}{\epsilon^2} + \frac{77}{18}\frac{1}{\epsilon} + 16 - \frac{11}{12}\pi^2 + \zeta(3)\right) + 2\,\ln^2\frac{\bar{s}_{pr}}{s}\left(-\frac{5}{6}\frac{1}{\epsilon} - \frac{77}{18}\right) - \frac{20}{18}\,\ln^3\frac{\bar{s}_{pr}}{s} \\
I^{\rm hcc}_{C_F q} &= \frac{1}{6}\frac{1}{\epsilon^2} + \left(\frac{13}{36} + \frac{\pi^2}{9}\right)\frac{1}{\epsilon} - \frac{119}{216} + \frac{17}{108}\,\pi^2 + \frac{14}{3}\,\zeta(3) \\
&\quad + 2\,\ln\frac{\bar{s}_{pr}}{s}\left(-\frac{1}{6}\frac{1}{\epsilon} - \frac{13}{36} - \frac{\pi^2}{9}\right) + \frac{1}{3}\,\ln^2\frac{\bar{s}_{pr}}{s} \\
I^{\rm hcc}_{C_F^2} &= -\frac{2}{\epsilon^3} - \frac{37}{4}\frac{1}{\epsilon^2} + \left(-\frac{333}{8} + \frac{7}{2}\,\pi^2 - 6\,\zeta(3)\right)\frac{1}{\epsilon} - \frac{2815}{16} + \frac{127}{8}\,\pi^2 + \frac{187}{3}\,\zeta(3) - \frac{31}{60}\,\pi^4 \\
&\quad + 2\,\ln\frac{\bar{s}_{pr}}{s}\left(\frac{2}{\epsilon^2} + \frac{37}{4}\frac{1}{\epsilon} + \frac{333}{8} - \frac{7}{2}\,\pi^2 + 6\,\zeta(3)\right) + 2\,\ln^2\frac{\bar{s}_{pr}}{s}\left(-\frac{2}{\epsilon} - \frac{37}{4}\right) + \frac{8}{3}\,\ln^3\frac{\bar{s}_{pr}}{s} \\
I^{\rm hcc}_{C_F C_A} &= -\frac{1}{2}\frac{1}{\epsilon^3} - \frac{23}{12}\frac{1}{\epsilon^2} + \left(-\frac{365}{72} - \frac{7}{36}\,\pi^2 + 5\,\zeta(3)\right)\frac{1}{\epsilon} - \frac{3089}{432} - \frac{163}{216}\,\pi^2 - \frac{49}{3}\,\zeta(3) + \frac{53}{120}\,\pi^4 \\
&\quad + 2\,\ln\frac{\bar{s}_{pr}}{s}\left(\frac{1}{2}\frac{1}{\epsilon^2} + \frac{23}{12}\frac{1}{\epsilon} + \frac{365}{72} + \frac{7}{36}\,\pi^2 - 5\,\zeta(3)\right) + 2\,\ln^2\frac{\bar{s}_{pr}}{s}\left(-\frac{1}{2}\frac{1}{\epsilon} - \frac{23}{12}\right) + \frac{2}{3}\,\ln^3\frac{\bar{s}_{pr}}{s}
\end{aligned}
$$

Proof-of-concept example

- $T_R C_F$ contribution to $\sigma_{\rm NNLO}(e^+e^- \to q\bar{q})$



- Finiteness in the n-body phase space:

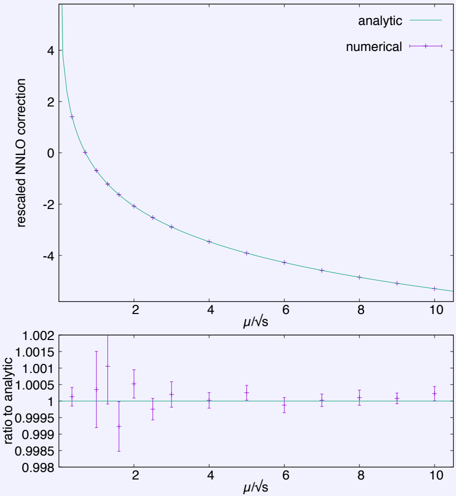
$$VV + I^{(2)} + I^{(\mathbf{RV})} \quad = \quad B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{8}{3}\zeta_3 - \frac{1}{9}\pi^2 - \frac{44}{9} - \frac{4}{3}\ln\frac{\mu^2}{s}\right).$$

- Finiteness in the $(n+1)$-body phase space, sector by sector:

$$RV\,\overline{\mathcal{W}}_{hq} + I^{(\mathbf{1})}_{hq} \quad = \quad -\frac{\alpha_S}{2\pi}\,\frac{2}{3}\,T_R\left(\ln\frac{\mu^2}{\bar{s}_{[34]r}} + \frac{8}{3}\right) R\,\overline{\mathcal{W}}_{hq}.$$

$$\overline{K}^{(\mathbf{RV})}_{hq} - I^{(\mathbf{12})}_{hq} \quad = \quad -\frac{\alpha_S}{2\pi}\,\frac{2}{3}\,T_R\left(\ln\frac{\mu^2}{\bar{s}_{[34]r}} + \frac{8}{3}\right)\left[\bar{\mathbf{S}}_h + \bar{\mathbf{C}}_{hq}\left(1 - \bar{\mathbf{S}}_h\right)\right] R\,\overline{\mathcal{W}}_{hq}\,.$$

Total NNLO cross section



- Example for $\mu/\sqrt{s} = 0.35$.
- Analytic:
  $\sigma_{\text{NNLO}} = \sigma_{\text{LO}}\, k\ \times 1.40787186$
- Subtraction method:
  $\sigma_{\text{NNLO}} = \sigma_{\text{LO}}\, k\ \times (1.40806 \pm 0.00040)$
- $k = \left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F$

NNLO summary

- Sector functions at NNLO engineered to factorise structure of NLO partitions.
- Sector-function sum rules allow analytic integration of double-unresolved counterterm.
- Todo: combine results to show analytic cancellation of $1/\epsilon$ poles for generic process (final-state radiation, massless).
- Todo: efficient numerical code.

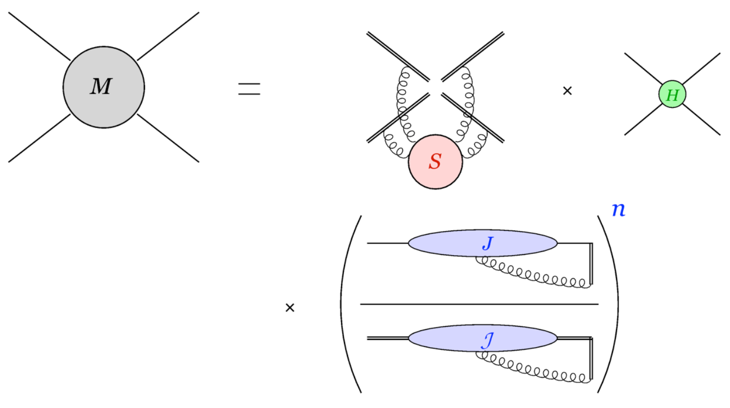
Factorisation and subtraction beyond NLO

Linking factorisation and subtraction

- The infrared structure of virtual corrections to gauge amplitudes is very well understood.
- Factorisation of virtual corrections contains all-order information.
  - Exponentiation of virtual corrections tightly connects high orders to low orders.
  - Classes of possible virtual poles are absent (e.g. massless tripoles).
  - Factorisation compactly encodes removal of overlapping soft-collinear poles.
- Structure of virtual poles must reflect in real singularities, as they add up to finite xsec.
- Use virtual structure as a principle to organise real subtraction counterterms beyond NLO.

Virtual-amplitude factorisation

$$\mathcal{M}_n\left(\frac{p_i}{\mu}\right) = \prod_{i=1}^{n} \left[\frac{\mathcal{J}_i\Big((p_i \cdot n_i)^2/(n_i^2 \mu^2)\Big)}{\mathcal{J}_{i,\,\mathrm{E}}\Big((\beta_i \cdot n_i)^2/n_i^2\Big)} \right] \mathcal{S}_n\left(\beta_i \cdot \beta_j\right) \mathcal{H}_n\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}\right)$$

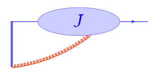


Definitions of soft, jets, eikonal jets

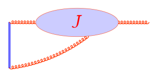
$\beta_i = p_i/\mu$, $n_i^2 \neq 0$.

Wilson lines $\Phi_v(\lambda_2,\lambda_1) \equiv \mathcal{P} \exp\left[\mathrm{i} g_s \int_{\lambda_1}^{\lambda_2} d\lambda \, v \cdot A(\lambda v) \right]$.

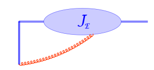
$$\overline{u}_s(p)\, \mathcal{J}_q\left(\frac{(p\cdot n)^2}{n^2\mu^2}\right) = \langle p, s \,|\, \overline{\psi}(0)\, \Phi_n(0,\infty)\, |0\rangle$$



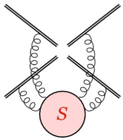
$$g_s\, \varepsilon_\mu^{*\,(\lambda)}(k) \mathcal{J}_g^{\mu\nu}\left(\frac{(k\cdot n)^2}{n^2\mu^2}\right) \equiv \langle k, \lambda | \left[\Phi_n(\infty,0)\, \mathrm{i} D^\nu\, \Phi_n(0,\infty) \right] |0\rangle$$



$$\mathcal{J}_{\mathrm{E}}\left(\frac{(\beta\cdot n)^2}{n^2}\right) = \langle 0 |\, \Phi_\beta(\infty,0)\, \Phi_n(0,\infty)\, |0\rangle$$

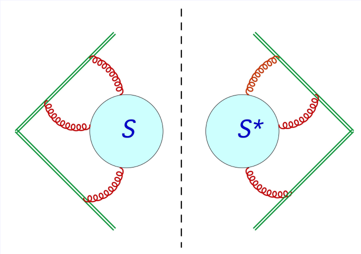


$$\mathcal{S}_n\left(\beta_i\cdot\beta_j\right) = \langle 0 | \prod_{k=1}^{n} \Phi_{\beta_k}(\infty,0)\, |0\rangle$$

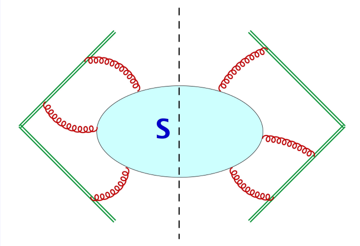


Real counterterms from virtual factorisation

- Consider for instance soft-only singularities.
- Fully inclusive virtual (left) + real (right) cross section is finite.



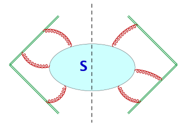
+



= finite

- Left frame contains virtual soft poles that cancel real soft singularities on the right.
- Define real soft counterterms as the right cut blob,
  i.e. Wilson lines between vacuum and a physical state with m soft partons.
- Analogously for collinear.

Soft currents to all orders



(picture for $n = 2$)

$$= \sum_{\{\lambda_i\}} \langle 0| \prod_{i=1}^{n} \Phi_{\beta_i}(0,\infty) |k_1,\lambda_1;\ldots;k_m,\lambda_m\rangle \langle k_1,\lambda_1;\ldots;k_m,\lambda_m| \prod_{i=1}^{n} \Phi_{\beta_i}(\infty,0) |0\rangle \equiv S_{n,\,m}$$

- $S_{n,m}$, radiative soft function: m soft partons emitted from n hard ones. Gauge invariant and containing loop corrections to all orders (α_S^{ℓ}).
- Generating the whole tower of real soft singularities. E.g. $(m,\ell) = (2,0) \Leftrightarrow \mathbf{S}_{ij}\, RR$ (NNLO); $(m,\ell) = (1,1) \Leftrightarrow \mathbf{S}_{i}\, RV$ (NNLO); $(m,\ell) = (2,1) \Leftrightarrow \mathbf{S}_{ij}\, RRV$ (N^3LO); ...
- Soft finiteness ensured by completeness relation

$$\sum_{m=0}^{\infty} \int d\Phi_m \, S_{n,m}(k_1,\ldots,k_m;\beta_i) = \langle 0| \prod_{i=1}^{n} \Phi_{\beta_i}(0,\infty) \prod_{i=1}^{n} \Phi_{\beta_i}(\infty,0) |0\rangle = \text{finite}.$$

Application: organisation of soft currents

- Radiative-amplituide factorisation (assume no collinear singularities)

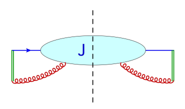
$$\mathcal{M}_{n,m}(k_1,\ldots,k_m,p_i) = \mathcal{S}_{n,m}(k_1,\ldots,k_m,\beta_i)\,\mathcal{H}_n(p_i) + \text{finite}\,.$$

- Letting $\mathcal{M}_{n,1} = \epsilon \cdot J_{\rm soft}\, \mathcal{M}_{n,0}$, the k-loop soft current for one radiation easily written.
- 0, 1, 2 loops:

$$\begin{aligned}
\epsilon \cdot J^{(0)}_{\rm soft} &= \mathcal{S}^{(0)}_{n,\,1}\,(k;\beta_i) = \epsilon^{*\,(\lambda)}_{\mu}(k)\, g_s \sum_{i=1}^{n} \frac{\beta_i^\mu}{\beta_i\cdot k}\,\mathbf{T}_i\,,\\
\epsilon \cdot J^{(1)}_{\rm soft} &= \mathcal{S}^{(1)}_{n,\,1}\,(k;\beta_i) - \mathcal{S}^{(0)}_{n,\,1}\,(k;\beta_i)\; \mathcal{S}^{(1)}_{n,\,0}\,(\beta_i)\\
\epsilon \cdot J^{(2)}_{\rm soft} &= \mathcal{S}^{(2)}_{n,\,1}\,(k;\beta_i) - \mathcal{S}^{(1)}_{n,\,1}\,(k;\beta_i)\; \mathcal{S}^{(1)}_{n,\,0}\,(\beta_i) - \mathcal{S}^{(0)}_{n,\,1}\,(\beta_i)\left[\mathcal{S}^{(2)}_{n,\,0}\,(\beta_i) - (\mathcal{S}^{(1)}_{n,\,0}\,(\beta_i))^2\right],\\
&\ldots
\end{aligned}$$

- Results for $J^{(0)}_{\rm soft}$ and $J^{(1)}_{\rm soft}$ reproduce all known results [Bassetto, Ciafaloni, Marchesini, 1984], [Berends, Giele 1989], [Catani, Grazzini, 2000].
- So far $J^{(2)}_{\rm soft}$ computed for 2 coloured legs by taking soft limit of full matrix elements [Badger, Glover, 2004] at $\mathcal{O}(\epsilon^0)$, [Gehrmann, Duhr, 2013] at $\mathcal{O}(\epsilon^2)$.
- Calculations based on eikonal Feynman rules, convenient to simplify/systematise these calculations (e.g. Feynman gauge vs axial gauge).

Collinear kernels to all orders



$$= \int d^d x \, \mathrm{e}^{\mathrm{i} l \cdot x} \sum_{\{\lambda_j\}} \langle 0 | \, \Phi_n(\infty, x) \, \psi(x) | p, s; k_j, \lambda_j \rangle \, \langle p, s; k_j, \lambda_j | \overline{\psi}(0) \, \Phi_n(0, \infty) \, | 0 \rangle \; \equiv \; J_{q,\, m}$$

- $J_{q,\, m}$ radiative jet function: m partons collinear to q (or g).
  Gauge invariant and containing loop corrections to all orders (α_S^ℓ).

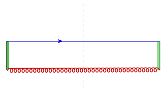
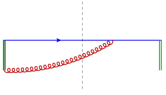
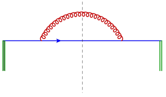
- Generating the whole tower of real collinear singularities.
  E.g. $(m, \ell) = (2, 0) \Leftrightarrow \mathbf{C}_{ijq}\, RR$ (NNLO); $(m, \ell) = (1, 1) \Leftrightarrow \mathbf{C}_{iq}\, RV$ (NNLO);
  $(m, \ell) = (2, 1) \Leftrightarrow \mathbf{C}_{ijq}\, RRV$ (N^{3}LO); ...

- Collinear finiteness ensured by completeness relation

$$\sum_{m=0}^{\infty} \int d\Phi_{m+1} \, J_{q,\, m} \, (k_1, \ldots, k_m; l, p, n) \; = \; \mathrm{Disc} \int d^d x \, \mathrm{e}^{\mathrm{i} l \cdot x} \, \langle 0 | \, \Phi_n(\infty, x) \psi(x) \overline{\psi}(0) \Phi_n(0, \infty) \, | 0 \rangle \; .$$

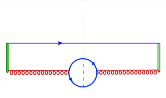
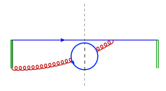
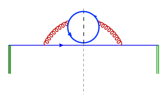
Application: calculation of multiple-collinear kernels

- At NLO, Feynman-gauge computation:

$\lim_{l_\perp \to 0} \Bigg($  $+$  $+ \text{h.c} +$  $\Bigg)$

$$= \lim_{l_\perp \to 0} \sum_s J_{q,\,1}\,(k; l, p, n) \;=\; \frac{8\pi\alpha_S}{l^2}\,(2\pi)^d\,\delta^d(l-p-k)\,P^{(0)}_{q\to qg}(z)$$

- At NNLO:

$\lim_{l_\perp \to 0} \Bigg($  $+$  $+ \text{h.c} +$  $\Bigg)$

$$= \lim_{l_\perp \to 0} \sum_s J_{q,\,2}\,(k_1, k_2; l, p, n) \;=\; \left(\frac{8\pi\alpha_S}{l^2}\right)^2 (2\pi)^d\,\delta^d(l-p-k_1-k_2)\,P^{(0)}_{q\to qq'q'}(z_1, z_2)$$

- Information on polarisations retained, full azimuthal kernels if ancestor parton is a gluon.

Organisation of counterterms for NLO subtraction

- Disclaimer: not yet a subtraction (no remappings), but a path to compact organisation.

- Start from virtual factorisation ($\mathcal{H}_n^{(0)} = \mathcal{M}_n^{(0)}$)

$$V \;=\; \mathcal{H}_n^{(0)\,\dagger} S_{n,\,0}^{(1)} \mathcal{H}_n^{(0)} + \sum_{i=1}^{n} \mathcal{H}_n^{(0)\,\dagger} \left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)} \right) \mathcal{H}_n^{(0)} + \text{finite}\,.$$

- Write down completeness relations for all virtual structures that appear

$$S_{n,0}^{(1)} + \int d\Phi_1 \, S_{n,\,1}^{(0)} \;=\; \text{finite}\,, \qquad \left(J_{i,\,0}^{(1)} - J_{i,\,E,\,0}^{(1)} \right) + \int d\Phi_1 \left(J_{i,\,1}^{(0)} - J_{i,\,E,\,1}^{(0)} \right) \;=\; \text{finite}\,,$$

- Define real counterterms as the integrands in the completeness relations

$$K^{\mathrm{s}} \;=\; \mathcal{H}_n^{(0)\,\dagger} S_{n,\,1}^{(0)} \mathcal{H}_n^{(0)}\,,$$

$$K^{\mathrm{hc}} \;=\; \sum_{i=1}^{n} \mathcal{H}_n^{(0)\,\dagger} \left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)} \right) \mathcal{H}_n^{(0)}\,.$$

Organisation of counterterms for NNLO subtraction

- Start from virtual factorisation, organise poles according to their physical origin

$$
\begin{aligned}
VV &\equiv VV^{(2\mathrm{s})} + VV^{(1\mathrm{s})} + \sum_{i=1}^{n} VV_i^{(2\mathrm{hc})} + \dots \\
VV^{(2\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,0}^{(2)}\, \mathcal{H}_n^{(0)}\,, \\
VV^{(1\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger} S_{n,\,0}^{(1)} \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger} S_{n,\,0}^{(1)} \mathcal{H}_n^{(0)}\,, \\
VV_i^{(2\mathrm{hc})} &= \mathcal{H}_n^{(0)\,\dagger} \Big[J_{i,\,0}^{(2)} - J_{i,\,\mathrm{E},\,0}^{(2)} - J_{i,\,\mathrm{E},\,0}^{(1)} \Big(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)} \Big) \Big] \mathcal{H}_n^{(0)}
\end{aligned}
$$

- Write down completeness relations for all virtual structures that appear

$$
\begin{aligned}
& S_{n,0}^{(2)} + \int d\Phi_1\, S_{n,\,1}^{(1)} + \int d\Phi_2\, S_{n,\,2}^{(0)} = \mathrm{finite}\,, \\
& J_{i,\,0}^{(2)} + \int d\Phi_1\, J_{i,\,1}^{(1)} + \int d\Phi_2\, J_{i,\,2}^{(0)} = \mathrm{finite}\,, \\
& \Big[J_{i,\,\mathrm{E},0}^{(1)} + \int d\Phi_1\, J_{i,\,\mathrm{E},\,1}^{(0)} \Big] \Big[J_{i,0}^{(1)} - J_{i,\,\mathrm{E},0}^{(1)} + \int d\Phi_1' \Big(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},1}^{(0)} \Big) \Big] = \mathrm{finite}\,.
\end{aligned}
$$

- Relations linking double-virtual, real-virtual, and double-real singularities: tower of counterterms defined from virtual poles through completeness.

Organisation of counterterms for NNLO subtraction

$$
\begin{aligned}
K^{(2\mathrm{s})}_{n+2} &= \mathcal{H}^{(0)\,\dagger}_{n}\, S^{(0)}_{n,\,2}\, \mathcal{H}^{(0)}_{n}\,. \\
K^{(2\mathrm{hc})}_{n+2,\,i} &= \mathcal{H}^{(0)\,\dagger}_{n} \left[J^{(0)}_{i,\,2} - J^{(0)}_{i,\,\mathrm{E},\,2} - J^{(0)}_{i,\,\mathrm{E},\,1} \left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1} \right) \right] \mathcal{H}^{(0)}_{n}\,, \\
K^{(2\mathrm{hc})}_{n+2,\,ij} &= \mathcal{H}^{(0)\,\dagger}_{n} \left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1} \right) \left(J^{(0)}_{j,\,1} - J^{(0)}_{j,\,\mathrm{E},\,1} \right) \mathcal{H}^{(0)}_{n} \\
K^{(1\mathrm{hc},\,1\mathrm{s})}_{n+2,\,i} &= \mathcal{H}^{(0)\,\dagger}_{n} \left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1} \right) S^{(0)}_{n,\,1} \mathcal{H}^{(0)}_{n}
\end{aligned}
$$

$$
\begin{aligned}
K^{(\mathbf{1},\,\mathrm{s})}_{n+2} &= \mathcal{H}^{(0)\,\dagger}_{n+1}\, S^{(0)}_{n+1,\,1}\, \mathcal{H}^{(0)}_{n+1}\,. \\
K^{(\mathbf{1},\,\mathrm{hc})}_{n+2,\,i} &= \mathcal{H}^{(0)}_{n+1}{}^{\dagger} \left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1} \right) \mathcal{H}^{(0)}_{n+1}\,,
\end{aligned}
$$

$$
\begin{aligned}
K^{(\mathbf{RV},\,\mathrm{s})}_{n+1} &= \mathcal{H}^{(0)\,\dagger}_{n}\, S^{(0)}_{n,\,1}\, \mathcal{H}^{(1)}_{n} + \mathcal{H}^{(1)\,\dagger}_{n}\, S^{(0)}_{n,\,1}\, \mathcal{H}^{(0)}_{n} + \mathcal{H}^{(0)\,\dagger}_{n}\, S^{(1)}_{n,\,1}\, \mathcal{H}^{(0)}_{n}\,. \\
K^{(\mathbf{RV},\,\mathrm{hc})}_{n+1,\,i} &= \mathcal{H}^{(0)\,\dagger}_{n} \left[J^{(1)}_{i,\,1} - J^{(1)}_{i,\,\mathrm{E},\,1} - J^{(1)}_{i,\,0} J^{(0)}_{i,\,\mathrm{E},\,1} - J^{(1)}_{i,\,\mathrm{E},\,0} J^{(0)}_{i,\,1} + 2 J^{(1)}_{i,\,\mathrm{E},\,0} J^{(0)}_{i,\,\mathrm{E},\,1} \right] \mathcal{H}^{(0)}_{n}\,. \\
K^{(\mathbf{RV},\,1\mathrm{hc},\,1\mathrm{s})}_{n+1,\,i} &= \left(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0} \right) \mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,1} \mathcal{H}^{(0)}_{n} + \left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1} \right) \mathcal{H}^{(0)\,\dagger}_{n} S^{(1)}_{n,\,0} \mathcal{H}^{(0)}_{n}\,, \\
K^{(\mathbf{RV},\,1\mathrm{hc})}_{n+1,\,i} &= \left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1} \right) \left(\mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,0} \mathcal{H}^{(1)}_{n} + \mathcal{H}^{(1)\,\dagger}_{n} S^{(0)}_{n,\,0} \mathcal{H}^{(0)}_{n} \right)\,. \\
K^{(\mathbf{RV},\,\mathrm{hc})}_{n+1,\,ij} &= \mathcal{H}^{(0)\,\dagger}_{n} \left[\left(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0} \right) \left(J^{(0)}_{j,\,1} - J^{(0)}_{j,\,\mathrm{E},\,1} \right) + (i \leftrightarrow j) \right] \mathcal{H}^{(0)}_{n}\,,
\end{aligned}
$$

Outlook

Outlook

- A new method for subtraction at NNLO, applied so far to QCD FSR massless only.
  - Phase space partitioned into sectors to minimise complexity (FKS sectors at NNLO).
  - Phase-space mappings adapted to ease analytic integration (CS mappings at NNLO).
  - Fully analytic integration of all counterterms achieved.
  - Ongoing general subtraction formula and planned extensions to ISR.

Outlook

- A new method for subtraction at NNLO, applied so far to QCD FSR massless only.
  - Phase space partitioned into sectors to minimise complexity (FKS sectors at NNLO).
  - Phase-space mappings adapted to ease analytic integration (CS mappings at NNLO).
  - Fully analytic integration of all counterterms achieved.
  - Ongoing general subtraction formula and planned extensions to ISR.
- Links between subtraction and factorisation.
  - Exploit all-order structure of virtual poles to organise real counterterms beyond N(N)LO.
  - Cancellation of singularities encoded through completeness relations.
  - Transparent structure of singularities, potential simplifications for higher-order kernels.

Thank you

Backup

Soft/collinear commutation at NLO

- Soft limit $\mathbf{S}_i$ ($k_i^\mu \to 0$): $s_{ia}/s_{ib} \to$ constant, $s_{ia}/s_{bc} \to 0, \forall\, a, b, c \neq i$.
- Collinear limit $\mathbf{C}_{ij}$ ($k_\perp \to 0$): $s_{ij}/s_{ia} \to 0$, $s_{ij}/s_{jb} \to 0$, $s_{ij}/s_{ab} \to 0, \forall\, a, b \neq i, j$. $s_{ia}/s_{ja} \to$ independent of a.
- Commutation in case i = gluon and j = quark.
- Altarelli-Parisi collinear kernel involved is $P_{ij}(x_i) = [1 + (1 - x_i)^2]/x_i$, with $x_i = s_{ir}/(s_{ir} + s_{jr})$, with arbitrary $r \neq i, j$.

$$
\begin{aligned}
\mathbf{S}_i\, R &= -\mathcal{N}_1 \sum_{\substack{l\neq i\\ m\neq i}} \frac{s_{lm}}{s_{il} s_{im}} B_{lm} \\
\Longrightarrow \quad \mathbf{C}_{ij}\, \mathbf{S}_i\, R &= -2\, \mathcal{N}_1 \sum_{l\neq i,j} \mathbf{C}_{ij} \frac{s_{jl}}{s_{il} s_{ij}} B_{lj} \;=\; -\,2\, \mathcal{N}_1 \frac{s_{jr}}{s_{ir} s_{ij}} (-C_{f_j} B), \\
\mathbf{C}_{ij}\, R &= \mathcal{N}_1 \frac{1}{s_{ij}} C_{f_j} B \frac{1 + [1 - s_{ir}/(s_{ir} + s_{jr})]^2}{s_{ir}/(s_{ir} + s_{jr})} \\
\Longrightarrow \quad \mathbf{S}_i \mathbf{C}_{ij} R &= -2\, \mathcal{N}_1 \frac{s_{jr}}{s_{ir} s_{ij}} (-C_{f_j} B).
\end{aligned}
$$

Sector functions

- Example of sector functions at NLO ($s_{qi} = 2\, q_{\rm cm} \cdot k_i, \quad s_{ij} = 2\, k_i \cdot k_j$), similar to those used in `MadFKS` [Frederix, et al., 0908.4272]:

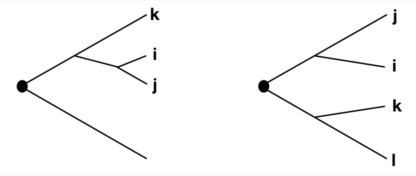
$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{k,\, l \neq k} \sigma_{kl}}\,, \qquad \text{with} \qquad \sigma_{ij} = \frac{1}{e_i\, w_{ij}}\,, \qquad e_i = \frac{s_{qi}}{s}\,, \qquad w_{ij} = \frac{s\, s_{ij}}{s_{qi}\, s_{qj}}\,.$$

- Example of sector functions at NNLO:

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sum_{a,\, b \neq a} \sum_{\substack{c \neq a \\ d \neq a,c}} \sigma_{abcd}}\,, \qquad \sigma_{ijkl} = \frac{1}{e_i^{\alpha}\, w_{ij}^{\beta}} \frac{1}{(e_k + \delta_{kj}\, e_i)\, w_{kl}}\,, \qquad \alpha > \beta > 1\,.$$

- Allowed index combinations: $\mathcal{W}_{ijjk}$, $\mathcal{W}_{ijkj}$, $\mathcal{W}_{ijkl}$.

- Roughly, sector functions select singularities relevant to stwo topologies (left: $\mathcal{W}_{ijjk}$, $\mathcal{W}_{ijkj}$, right: $\mathcal{W}_{ijkl}$)



Soft counterterm in FKS

- The soft FKS counterterm does not feature gluon energy: it reduces to an angular integral

$$I^{\rm s}_{\rm FKS} \propto \sum_{lm} \int d\cos\theta \, d\phi (\sin\phi \sin\theta)^{-2\epsilon} \frac{1-\cos\theta_{lm}}{(1-\cos\theta_{li})(1-\cos\theta_{mi})} .$$

- Doable (actually relevant to angular-ordering), but not maximally easy: relations among θ_{lm}, θ_{li} and θ_{mi} are non-trivial in terms of integration variables.

- Analogous features at NNLO may be much more severe.

Cancellation of virtual NLO poles

- Integrated counterterm I computed at all orders in ϵ.
- ϵ expansion:

$$
\begin{aligned}
I(\{\bar{k}\}) \;=\; & \frac{\alpha_S}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\Bigg\{\Bigg[B(\{\bar{k}\})\sum_k\left(\frac{C_{f_k}}{\epsilon^2}+\frac{\gamma_k}{\epsilon}\right)+\sum_{k,\,l\neq k}B_{kl}(\{\bar{k}\})\,\frac{1}{\epsilon}\ln\bar{\eta}_{kl}\Bigg] \\
& + \Bigg[B(\{\bar{k}\})\sum_k\Bigg(\delta_{f_k g}\,\frac{C_A+4\,T_R\,N_f}{6}\left(\ln\bar{\eta}_{kr}-\frac{8}{3}\right) \\
& + \delta_{f_k g}\,C_A\left(6-\frac{7}{2}\zeta_2\right)+\delta_{f_k\{q,\bar{q}\}}\,\frac{C_F}{2}\left(10-7\zeta_2+\ln\bar{\eta}_{kr}\right)\Bigg) \\
& + \sum_{k,\,l\neq k}B_{kl}\left(\{\bar{k}\}\right)\,\ln\bar{\eta}_{kl}\left(2-\frac{1}{2}\ln\bar{\eta}_{kl}\right)\Bigg]\Bigg\}\,.
\end{aligned}
$$

- $\bar{\eta}_{ab}=\bar{s}_{ab}/s$, and $\gamma_k \;=\; \delta_{f_k g}\,\frac{11C_A-4\,T_R N_f}{6}\;+\;\delta_{f_k\{q,\bar{q}\}}\,\frac{3}{2}C_F$.
- Same structure of ϵ singularities as V (up to a sign).

NNLO sector-function sum rules

$$\mathbf{S}_{ik}\left(\sum_{b\neq i}\sum_{d\neq i,k}\mathcal{W}_{ibkd}+\sum_{b\neq k}\sum_{d\neq k,i}\mathcal{W}_{kbid}\right)=1\,,$$

$$\mathbf{C}_{ijk}\sum_{abc\,\in\,\mathrm{perm}(ijk)}\left(\mathcal{W}_{abbc}+\mathcal{W}_{abcb}\right)=1\,,$$

$$\mathbf{S}_{i}\,\mathbf{C}_{ijk}\left(\mathcal{W}_{ij}^{(\alpha\beta)}+\mathcal{W}_{ik}^{(\alpha\beta)}\right)=1\,,$$

$$\mathbf{S}_{ij}\,\mathbf{C}_{ijk}\sum_{ab\,\in\,\mathrm{perm}(ij)}\left(\mathcal{W}_{abbk}+\mathcal{W}_{akbk}\right)=1\,,\qquad \mathbf{S}_{ik}\,\mathbf{C}_{ijkl}\left(\mathcal{W}_{ijkl}+\mathcal{W}_{klij}\right)=1\,.$$

$$\mathbf{SC}_{ijk}\,\mathbf{S}_{ij}\sum_{b\neq i}\mathcal{W}_{ibjk}=1\,,\qquad \mathbf{CS}_{ijk}\,\mathbf{S}_{ik}\sum_{d\neq i,k}\mathcal{W}_{ijkd}=1\,,$$

$$\mathbf{CS}_{ijk}\,\mathbf{C}_{ijk}\left(\mathcal{W}_{ijkj}+\mathcal{W}_{jiki}\right)=1\,,\qquad \mathbf{CS}_{ijk}\,\mathbf{C}_{ijkl}\left(\mathcal{W}_{ijkl}+\mathcal{W}_{jikl}\right)=1\,,$$

$$\mathbf{CS}_{ijk}\,\mathbf{C}_{ijk}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkj}=1\,,\qquad \mathbf{CS}_{ijk}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkl}=1\,,$$

$$\mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\sum_{ab\,\in\,\mathrm{perm}(jk)}\left(\mathcal{W}_{iaab}+\mathcal{W}_{iaba}\right)=1\,,\qquad \mathbf{SC}_{ikl}\,\mathbf{C}_{ijkl}\left(\mathcal{W}_{ijkl}+\mathcal{W}_{ijlk}\right)=1\,,$$

$$\mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\,\mathbf{S}_{ik}\left(\mathcal{W}_{ijkj}+\mathcal{W}_{ikkj}\right)=1\,,\qquad \mathbf{SC}_{ijk}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkl}=1\,.$$

Double-radiation phase space

- Catani-Seymour variables $y, z, y', z', x' \in [0,1]$ for mapping $\{k\} \to \{\bar{k}\}^{(abcd)}$:

$$
\begin{aligned}
s_{ab} &= y'\, y\, s_{abcd}\,, & s_{cd} &= (1-y')\,(1-y)\,(1-z)\; s_{abcd}\,,\\
s_{ac} &= z'\,(1-y')\; y\, s_{abcd}\,, & s_{bc} &= (1-y')\,(1-z')\; y\, s_{abcd}\,,
\end{aligned}
$$

$$
s_{ad} = (1-y)\left[y'\,(1-z')\,(1-z) + z'z - 2\,(1-2x')\,\sqrt{y'z'\,(1-z')\;z\,(1-z)}\right] s_{abcd}\,,
$$

$$
s_{bd} = (1-y)\left[y'z'\,(1-z) + (1-z')\,z + 2\,(1-2x')\,\sqrt{y'z'\,(1-z')\;z\,(1-z)}\right] s_{abcd}\,,
$$

- Phase-space factorisation:

$$
d\Phi_{n+2} \quad = \quad d\Phi_n^{(abcd)}\, d\Phi_{\mathrm{rad},2}^{(abcd)}\,,
$$

$$
\begin{aligned}
\int d\Phi_{\mathrm{rad},2}^{(abcd)} \quad &= \quad \int d\Phi_{\mathrm{rad},2}\left(s_{abcd};\, y, z, \phi, y', z', x'\right)\\
&= \quad N^2(\epsilon)\,(s_{abcd})^{2-2\epsilon}\int_0^1 dx' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\; (\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz\\
&\quad\; \times \left[4\,x'\,(1-x')\; y'\,(1-y')^2\; z'\,(1-z')\; y^2(1-y)^2\, z\,(1-z)\right]^{-\epsilon}\\
&\quad\; \times \left[x'\,(1-x')\right]^{-1/2}(1-y')\; y\,(1-y)\;.
\end{aligned}
$$

Integration of the double-unresolved part of $I^{(\mathbf{2})}$

$$I^{(2)} = \sum_{i,j>i} \int d\Phi_2 \, \overline{\mathbf{S}}_{ij} RR + \sum_{\substack{i,j>i \\ k>j}} \int d\Phi_2 \, \overline{\mathbf{C}}_{ijk} (1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}) RR + \ldots$$

Kernels to integrate

Catani, Grazzini
hep-ph: 9908523

2 soft kernels	$\mathcal{I}_{cd}^{(ij)}[q\bar{q}]$ $\mathcal{I}_{cd}^{(ij)}[gg]$
5 collinear kernels	$P_{ijk}[qq'\bar{q}']$ $P_{ijk}[qq\bar{q}]$ $P_{ijk}[q\bar{q}g]$ $P_{ijk}[qgg]$ $P_{ijk}[ggg]$

- Rational functions of six invariants s_{ab}, s_{ac}, s_{bc}, s_{cd}, s_{ad}, s_{bd}
- The possible denominators are only:

s_{ab}, s_{ac}, s_{bc}, s_{cd}, s_{ad}, s_{bd},

$s_{ac} + s_{bc}$, $s_{ad} + s_{bd}$, $s_{ad} + s_{cd}$, $s_{bd} + s_{cd}$

Integration of the double-unresolved part of $I^{(2)}$

- The integrala of the kernels are symmetric under:
  - the permutation of the four momenta k_a, k_c, k_b, k_d
  - the following permutations of invariants:

$$s_{ab} \leftrightarrow s_{cd} \qquad s_{ac} \leftrightarrow s_{bd} \qquad s_{ad} \leftrightarrow s_{bc}$$

- We can reduce the denominators to:

$$\begin{aligned}
s_{ab} &= y'\,y\,s_{abcd} \\
s_{ac} &= z'(1-y')\,y\,s_{abcd} \\
s_{bc} &= (1-y')(1-z')\,y\,s_{abcd} \\
s_{cd} &= (1-y')(1-y)(1-z)\,s_{abcd} \\
s_{bd} &= (1-y)\left[y'z'(1-z)+(1-z')z+2(1-2x')\sqrt{y'z'(1-z')z(1-z)}\right]s_{abcd} \\
s_{ac}+s_{bc} &= (1-y')\,y\,s_{abcd}, \\
s_{ad}+s_{bd} &= (y'+z-y'\,z)\,(1-y)\,s_{abcd}, \\
s_{ab}+s_{bc} &= (1-z'+z'y')\,y\,s_{abcd}.
\end{aligned}$$

- The integral measure is:

$$\begin{aligned}
\int d\Phi_2(p^2;y,z,\phi,y',z',x') &= G_2\,(p^2)^{2-2\epsilon}\int_0^1 dx'\int_0^1 dy'\int_0^1 dz'\int_0^1 dy\int_0^1 dz\,[x'(1-x')]^{-\epsilon-1/2} \\
&\quad \left[y'z'(1-y')^2(1-z')y^2z(1-y)^2(1-z)\right]^{-\epsilon}y(1-y)(1-y')
\end{aligned}$$

Integration of the double-unresolved part of $I^{(2)}$

- Using the properties of the hypergeometric function ${}_2F_1$, we are left with integrals of the following types:

$$\int_0^1 dt\,(1-t)^{\mu}\,t^{\nu}\;{}_2F_1(n_1, n_2-\epsilon, n_3-2\epsilon, 1-t)$$

$$\int_0^1 dt\int_0^1 du\,(1-t)^{\mu}\,t^{\nu}\,(1-u)^{\rho}\,u^{\sigma}\,{}_2F_1(n_1, n_2-\epsilon, n_3-2\epsilon, 1-tu)$$

$$n_1, n_2, n_3 \in \mathbb{N}, \qquad n_1 \geq 1, \qquad n_3 \geq n_1+1, n_2$$

$$\mu, \nu, \rho, \sigma = n + m\,\epsilon, \qquad n, m \in \mathbb{Z}, \qquad n \geq -1$$

- All integrals could be written in terms of the hypergeometric functions

$${}_2F_1(a,b,c,1), \qquad {}_3F_2(a,b,c,1) \qquad {}_4F_3(a,b,c,1)$$

and then expanded in ϵ

- We have expanded the ${}_2F_1$ in ϵ and then integrated in t and u

All integrals checked against a numerical computation without using symmetries

Integration of the double-unresolved part of $I^{(\mathbf{2})}$

Results for the integrated kernels

$$A = \frac{1}{(4\pi)^4}\left(\frac{s_{abcd}\,e^{\gamma_E}}{4\pi}\right)^{-2\epsilon}$$

$$\int d\Phi_2\, I^{(ij)}_{cd}[q\bar{q}] \;=\; A\left\{\frac{2}{3}\,\frac{1}{\epsilon^3}+\frac{28}{9}\,\frac{1}{\epsilon^2}+\left[\frac{416}{27}-\frac{7}{9}\pi^2\right]\frac{1}{\epsilon}+\frac{5260}{81}-\frac{104}{27}\pi^2-\frac{76}{9}\zeta(3)\right\} \qquad c\neq d$$

$$\int d\Phi_2\, I^{(ij)}_{cc}[q\bar{q}] \;=\; A\left\{-\frac{2}{3}\,\frac{1}{\epsilon^2}-\frac{16}{9}\,\frac{1}{\epsilon}-\frac{212}{27}+\pi^2\right\}$$

$$\int d\Phi_2\, I^{(ij)}_{cd}[gg] \;=\; A\left\{\frac{2}{\epsilon^4}+\frac{35}{3}\,\frac{1}{\epsilon^3}+\left[\frac{481}{9}-\frac{8}{3}\pi^2\right]\frac{1}{\epsilon^2}+\left[\frac{6218}{27}-\frac{269}{18}\pi^2-\frac{154}{3}\zeta(3)\right]\frac{1}{\epsilon}+\frac{76912}{81}-\frac{3775}{54}\pi^2-\frac{2050}{9}\zeta(3)-\frac{23}{60}\pi^4\right\}$$

$$\int d\Phi_2\, I^{(ij)}_{cc}[gg] \;=\; A\left\{-\frac{2}{3}\,\frac{1}{\epsilon^2}-\frac{10}{9}\,\frac{1}{\epsilon}-\frac{164}{27}+\pi^2\right\}$$

$$\int d\Phi_2\, P_{ijk}[qq'\bar{q}'] \;=\; A\left\{-\frac{1}{3}\,\frac{1}{\epsilon^3}-\frac{31}{18}\,\frac{1}{\epsilon^2}+\left[-\frac{889}{108}+\frac{\pi^2}{2}\right]\frac{1}{\epsilon}-\frac{23941}{648}+\frac{31}{12}\pi^2+\frac{80}{9}\zeta(3)\right\}$$

$$\int d\Phi_2\, P_{ijk}[qq\bar{q}] \;=\; A\left\{\left[-\frac{13}{8}+\frac{1}{4}\pi^2-\zeta(3)\right]\frac{1}{\epsilon}-\frac{227}{16}+\pi^2+\frac{17}{2}\zeta(3)-\frac{11}{120}\pi^4\right\}$$

$$\int d\Phi_2\, P^{(\mathrm{ab})}_{ijk}[gq\bar{q}] \;=\; A\left\{-\frac{2}{3}\,\frac{1}{\epsilon^3}-\frac{31}{9}\,\frac{1}{\epsilon^2}+\left[-\frac{889}{54}+\pi^2\right]\frac{1}{\epsilon}-\frac{23833}{324}+\frac{31}{6}\pi^2+\frac{160}{9}\zeta(3)\right\}$$

$$\int d\Phi_2\, P^{(\mathrm{nab})}_{ijk}[gq\bar{q}] \;=\; A\left\{-\frac{2}{3}\,\frac{1}{\epsilon^3}-\frac{41}{12}\,\frac{1}{\epsilon^2}+\left[-\frac{1675}{108}+\frac{17}{18}\pi^2\right]\frac{1}{\epsilon}-\frac{5404}{81}+\frac{1063}{216}\pi^2+\frac{139}{9}\zeta(3)\right\}$$

$$\int d\Phi_2\, P^{(\mathrm{ab})}_{ijk}[ggq] \;=\; A\left\{\frac{2}{\epsilon^4}+\frac{7}{\epsilon^3}+\left[\frac{251}{8}-3\pi^2\right]\frac{1}{\epsilon^2}+\left[\frac{2125}{16}-\frac{21}{2}\pi^2-\frac{154}{3}\zeta(3)\right]\frac{1}{\epsilon}+\frac{17607}{32}-\frac{753}{16}\pi^2-\frac{548}{3}\zeta(3)+\frac{13}{20}\pi^4\right\}$$

$$\int d\Phi_2\, P^{(\mathrm{nab})}_{ijk}[ggq] \;=\; A\left\{\frac{1}{2}\,\frac{1}{\epsilon^4}+\frac{8}{3}\,\frac{1}{\epsilon^3}+\left[\frac{905}{72}-\frac{2}{3}\pi^2\right]\frac{1}{\epsilon^2}+\left[\frac{11773}{216}-\frac{89}{24}\pi^2-\frac{65}{6}\zeta(3)\right]\frac{1}{\epsilon}+\frac{295789}{1296}-\frac{845}{48}\pi^2-\frac{2191}{36}\zeta(3)+\frac{19}{240}\pi^4\right\}$$

$$\int d\Phi_2\, P_{ijk}[ggg] \;=\; A\left\{\frac{5}{2}\,\frac{1}{\epsilon^4}+\frac{21}{2}\,\frac{1}{\epsilon^3}+\left[\frac{853}{18}-\frac{11}{3}\pi^2\right]\frac{1}{\epsilon^2}+\left[\frac{5450}{27}-\frac{275}{18}\pi^2-\frac{188}{3}\zeta(3)\right]\frac{1}{\epsilon}+\frac{180739}{216}-\frac{1868}{27}\pi^2-\frac{1555}{6}\zeta(3)+\frac{41}{60}\pi^4\right\}$$

Matrix elements for the $T_R C_F$ contrib. to $e^+e^- \to q\bar{q}$ at NNLO

- Analytic matrix elements from [Hamberg, van Neerven, Matsuura, 1991], [Gehrmann De Ridder, Gehrmann, Glover, 0403057], [Ellis, Ross, Terrano, 1980]

$$\begin{aligned}
VV &= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R\, C_F \Bigg\{ \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[\frac{1}{3\epsilon^3} + \frac{14}{9\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{11}{18}\pi^2 + \frac{353}{54}\right) + \left(-\frac{26}{9}\zeta_3 - \frac{77}{27}\pi^2 + \frac{7541}{324}\right)\right] \\
&\quad + \left(\frac{\mu^2}{s}\right)^{\epsilon} \left[-\frac{4}{3\epsilon^3} - \frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{7}{9}\pi^2 - \frac{16}{3}\right) + \left(\frac{28}{9}\zeta_3 + \frac{7}{6}\pi^2 - \frac{32}{3}\right)\right] \Bigg\}, \\
\int d\Phi_{\rm rad}\, RV &= \frac{\alpha_S}{2\pi}\, \frac{1}{\epsilon}\, \frac{2}{3}\, T_R \int d\Phi_{\rm rad}\, R \\
&= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R\, C_F \left(\frac{\mu^2}{s}\right)^{\epsilon} \left[\frac{4}{3\epsilon^3} + \frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{7}{9}\pi^2 + \frac{19}{3}\right) + \left(-\frac{100}{9}\zeta_3 - \frac{7}{6}\pi^2 + \frac{109}{6}\right)\right], \\
\int d\Phi_{\rm rad,2}\, RR &= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R\, C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[-\frac{1}{3\epsilon^3} - \frac{14}{9\epsilon^2} + \frac{1}{\epsilon}\left(\frac{11}{18}\pi^2 - \frac{407}{54}\right) + \left(\frac{134}{9}\zeta_3 + \frac{77}{27}\pi^2 - \frac{11753}{324}\right)\right].
\end{aligned}$$

Integrated counterterms in the $T_R C_F$ contrib. to $e^+e^- \to q\bar{q}$ at NNLO

$$
\begin{aligned}
I^{(\mathbf{2})} &= \int d\Phi_{\mathrm{rad},2} \left[\overline{\mathbf{S}}_{34} + \overline{\mathbf{C}}_{134} \left(1 - \overline{\mathbf{S}}_{34}\right) + \overline{\mathbf{C}}_{234} \left(1 - \overline{\mathbf{S}}_{34}\right) \right] RR \\
&= B \left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[-\frac{1}{3\epsilon^3} - \frac{14}{9\epsilon^2} + \frac{1}{\epsilon}\left(\frac{11}{18}\pi^2 - \frac{425}{54}\right) \right. \\
&\qquad \left. + \left(\frac{122}{9}\zeta_3 + \frac{74}{27}\pi^2 - \frac{12149}{324}\right) \right] + \mathcal{O}(\epsilon).
\end{aligned}
$$

$$
I^{(\mathbf{1})}_{hq} = -\frac{\alpha_S}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon} \frac{2}{3}\, T_R \left(\frac{1}{\epsilon} - \ln \bar{\eta}_{[34]r} + \frac{8}{3}\right) R\,\overline{\mathcal{W}}_{hq} + \mathcal{O}(\epsilon)\,,
$$

$$
I^{(\mathbf{12})}_{hq} = \frac{\alpha_S}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon} \frac{2}{3}\, T_R \left(\frac{1}{\epsilon} - \ln \bar{\eta}_{[34]r} + \frac{8}{3}\right) \left[\bar{\mathbf{S}}_h + \overline{\mathbf{C}}_{hq}\left(1 - \bar{\mathbf{S}}_h\right)\right] R\,\overline{\mathcal{W}}_{hq} + \mathcal{O}(\epsilon)\,.
$$

$$
\begin{aligned}
I^{(\mathbf{RV})} &= \frac{\alpha_S}{2\pi}\,\frac{2}{3}\,\frac{1}{\epsilon}\, T_R \int d\Phi_{\mathrm{rad}} \left[\overline{\mathbf{S}}_{[34]} + \overline{\mathbf{C}}_{1[34]}\left(1 - \overline{\mathbf{S}}_{[34]}\right) + \overline{\mathbf{C}}_{2[34]}\left(1 - \overline{\mathbf{S}}_{[34]}\right)\right] R \\
&= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R\, C_F \left(\frac{\mu^2}{s}\right)^{\epsilon} \left[\frac{4}{3\epsilon^3} + \frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left(\frac{7}{9}\pi^2 - \frac{20}{3}\right) - \left(\frac{100}{9}\zeta_3 + \frac{7}{6}\pi^2 - 20\right)\right] + \mathcal{O}(\epsilon)\,,
\end{aligned}
$$

Soft currents vs radiative soft function $\mathcal{S}$

Standard soft factorisation (à la Catani-Grazzini)

$$\mathcal{M}^{(0)}_{n,1} = \epsilon \cdot J^{(0)}_{\rm soft}\, \mathcal{M}^{(0)}_{n,0}$$

$$\mathcal{M}^{(1)}_{n,1} = \epsilon \cdot J^{(0)}_{\rm soft}\, \mathcal{M}^{(1)}_{n,0} + \epsilon \cdot J^{(1)}_{\rm soft}\, \mathcal{M}^{(0)}_{n,0}$$

$$\mathcal{M}^{(2)}_{n,1} = \epsilon \cdot J^{(0)}_{\rm soft}\, \mathcal{M}^{(2)}_{n,0} + \epsilon \cdot J^{(1)}_{\rm soft}\, \mathcal{M}^{(1)}_{n,0} + \epsilon \cdot J^{(0)}_{\rm soft}\, \mathcal{M}^{(2)}_{n,0}$$

Getting virtual $\mathcal{H}$'s from virtual $\mathcal{M}$'s.

$$\mathcal{M}^{(0)}_{n,0} = \mathcal{H}^{(0)}_{n,0}$$

$$\mathcal{M}^{(1)}_{n,0} = \mathcal{H}^{(1)}_{n,0} + \mathcal{S}^{(1)}_{n,0}\, \mathcal{H}^{(0)}_{n,0} \qquad \Longrightarrow \qquad \mathcal{H}^{(1)}_{n,0} = \mathcal{M}^{(1)}_{n,0} - \mathcal{S}^{(1)}_{n,0}\, \mathcal{M}^{(0)}_{n,0}\,,$$

$$\mathcal{M}^{(2)}_{n,0} = \mathcal{H}^{(2)}_{n,0} + \mathcal{S}^{(1)}_{n,0}\, \mathcal{H}^{(1)}_{n,0} + \mathcal{S}^{(2)}_{n,0}\, \mathcal{H}^{(0)}_{n,0}$$

$$\Longrightarrow \qquad \mathcal{H}^{(2)}_{n,0} = \mathcal{M}^{(2)}_{n,0} - \mathcal{S}^{(1)}_{n,0} \left[\mathcal{M}^{(1)}_{n,0} - \mathcal{S}^{(1)}_{n,0}\, \mathcal{M}^{(0)}_{n,0} \right] - \mathcal{S}^{(2)}_{n,0}\, \mathcal{M}^{(0)}_{n,0}$$

Factorisation $\mathcal{M}_{n,m} = S_{n,m} \mathcal{H}_{n,0}$

$$\mathcal{M}^{(0)}_{n,1} = \mathcal{S}^{(0)}_{n,1}\, \mathcal{M}^{(0)}_{n,0}$$

$$\mathcal{M}^{(1)}_{n,1} = \mathcal{S}^{(0)}_{n,1}\, \mathcal{M}^{(1)}_{n,0} + \left[\mathcal{S}^{(1)}_{n,1} - \mathcal{S}^{(0)}_{n,1}\, \mathcal{S}^{(1)}_{n,0} \right] \mathcal{M}^{(0)}_{n,0}$$

$$\mathcal{M}^{(2)}_{n,1} = \mathcal{S}^{(0)}_{n,1}\, \mathcal{M}^{(2)}_{n,0} + \left[\mathcal{S}^{(1)}_{n,1} - \mathcal{S}^{(0)}_{n,1}\, \mathcal{S}^{(1)}_{n,0} \right] \mathcal{M}^{(1)}_{n,0}$$
$$+ \left[\mathcal{S}^{(2)}_{n,1} - \mathcal{S}^{(1)}_{n,1}\, \mathcal{S}^{(1)}_{n,0} - \mathcal{S}^{(0)}_{n,1} \left(\mathcal{S}^{(2)}_{n,0} - {\mathcal{S}^{(1)}_{n,0}}^2 \right) \right] \mathcal{M}^{(0)}_{n,0}$$